CHAPTER 3

TRANSFORMS

The term *transform* refers to a mathematical operation that takes a given function, called the *original* and returns a new function, referred to as the *image*. The transformation is often done by means of an integral or summation formula. Commonly used transforms are named after Laplace and Fourier. Transforms are frequently used to change a complicated problem into a simpler one. The simpler problem is then solved in the image domain; next, by using the inverse transform we obtain the solution in the original domain. A standard example is the use of the Laplace transform to solve a differential equation, or the z transform to solve a difference equation.

The theory of transforms has two principal aspects:

- examining the nature of the signals or sequences and
- solving LTI systems by transforming differential or difference equations into algebraic equations.

The concept of transforms is based upon suitable mapping of

- functions representing signals or sequences into new (complex) functions, which we call the *images*, and
- the set of differential and difference equations, describing systems under analysis, into algebraic equations in a new (complex) variable.

We investigate signals and systems by investigating the images or complex equations in new (complex) variables.

We seek a suitable transformation $\mathcal{T}$,and the inverse transformation $\mathcal{T}^{-1}$, such that $\mathcal{T}(x) = X$ and $\mathcal{T}^{-1}(X) = x$, which has to satisfy the four important properties:

1. **uniqueness**, $\mathcal{T}(x_1) = \mathcal{T}(x_2) \Leftrightarrow x_1 = x_2$;
2. **homogeneity**, $\mathcal{T}(Kx) = K\mathcal{T}(x)$, $\mathcal{T}^{-1}(KX) = K\mathcal{T}^{-1}(X)$, K is a constant;
3. **additivity**, $\mathcal{T}(x_1 + x_2) = \mathcal{T}(x_1) + \mathcal{T}(x_2)$, $\mathcal{T}^{-1}(X_1 + X_2) = \mathcal{T}^{-1}(X_1) + \mathcal{T}^{-1}(X_2)$;
4. **differentiating** and **differencing**, the operation of differentiating or differencing maps into the algebraic operation of multiplication.

The subsequent sections review the definition and the salient properties of the most important transforms required by the advanced filter design studied in this book.

For an in-depth study of transforms readers can consult excellent books such as references 5–7, 10, 11, 14, and 16–18.

3.1 PHASOR TRANSFORMATION

Sinusoidal waveforms play an important role in science and engineering. If we know the response of a linear time-invariant (LTI) system to any sinusoidal signal, we know, in principle, its response to any signal [3]. In this chapter we shall develop a method for calculating the response of LTI systems to sinusoidal inputs, which is based on the idea of representing a sinusoid of a given frequency by a complex quantity.

3.1.1 The Representation of a Sinusoid by a Phasor

A *sinusoidal signal* (also called a *sinusoid*) of angular frequency ω is any function of time t defined on $(-\infty < t < +\infty)$ and of the form

$$x(t) = X_{\mathrm{m}} \cos(\omega t + \xi) \tag{3.1}$$

where the real constants X_{m}, ω, and ξ are called the *amplitude*, the *angular frequency*, and the *phase* of the sinusoid, respectively. The amplitude is taken to be positive, $X_{\mathrm{m}} > 0$. The angular frequency is measured in rad/s and is given by

$$\omega = 2\pi f$$

where f is the *frequency* of the sinusoid measured in Hz.

The algebraic sum of any number of sinusoids of the same angular frequency, say ω, and of any number of their derivatives of any order is also a sinusoid of the same angular frequency ω.

Any sinusoid can be expressed in terms of the exponential function of complex argument

$$x(t) = X_{\mathrm{m}} \cos(\omega t + \xi) = \frac{1}{2} X_{\mathrm{m}}\, \mathrm{e}^{j(\omega t + \xi)} + \frac{1}{2} X_{\mathrm{m}}\, \mathrm{e}^{-j(\omega t + \xi)}$$

or, equivalently,

$$x(t) = \frac{1}{2} X_{\mathrm{m}}\, \mathrm{e}^{j\xi}\, \mathrm{e}^{j\omega t} + \frac{1}{2} X_{\mathrm{m}}\, \mathrm{e}^{-j\xi}\, \mathrm{e}^{-j\omega t}$$

and we may write

$$x(t) = \frac{1}{2}\left(X_{\mathrm{m}}\,\mathrm{e}^{j\xi}\right)\left(\mathrm{e}^{j\omega t}\right) + \frac{1}{2}\left(\left(X_{\mathrm{m}}\,\mathrm{e}^{j\xi}\right)\left(\mathrm{e}^{j\omega t}\right)\right)^{*}$$

where j designates the imaginary unit, $j = \sqrt{-1}$, and the asterisk denotes complex conjugate values.

A sinusoid with angular frequency ω is completely specified by its amplitude X_{m} and its phase ξ, leading to the idea of representing the sinusoid by the complex quantity

$$X = X_{\mathrm{m}}\,\mathrm{e}^{j\xi} \tag{3.2}$$

Given the complex number X and the angular frequency ω, we recover the sinusoid (3.1) as follows:

$$x(t) = \mathrm{Re}(X\,\mathrm{e}^{j\omega t}) \tag{3.3}$$

The complex quantity X is called the *phasor* representing the sinusoid (3.1).

The mapping of a sinusoid $x(t)$ into a complex quantity X can be viewed as a transformation that we shall call the *phasor transformation*. Formally, we can establish this transformation by an integral formula

$$X = \mathcal{P}(x(t)) = \frac{\omega}{\pi}\int_{0}^{2\pi/\omega} x(t)\,\mathrm{e}^{-j\omega t}\,\mathrm{d}t \tag{3.4}$$

and we can establish the *inverse phasor transformation* by

$$x(t) = \mathcal{P}^{-1}(X) = \mathrm{Re}(X\,\mathrm{e}^{j\omega t}) \tag{3.5}$$

We shall use the following notation to explicitly designate the angular frequency, ω, to which the phasor, X, has been associated:

$$X^{(\omega)} = \mathcal{P}_{\omega}(x(t)) = \frac{\omega}{\pi}\int_{0}^{2\pi/\omega} x(t)\,\mathrm{e}^{-j\omega t}\,\mathrm{d}t \tag{3.6}$$

and

$$x(t) = \mathcal{P}_{\omega}^{-1}(X) = \mathrm{Re}(X\,\mathrm{e}^{j\omega t}) \tag{3.7}$$

When performing calculations with phasors we assume that the angular frequency is known and that it is the same for all phasors; that is, the operators $\mathcal{P}$ and $\mathcal{P}^{-1}$ assume the notion of ω.

Unless otherwise stated, the symbol t will always represent time.

3.1.2 Properties of the Phasor Transform

The phasor transform is unique for all t:

$$x_1(t) = x_2(t) \Leftrightarrow X_1 = X_2 \tag{3.8}$$

where $X_1 = \mathcal{P}(x_1(t))$, $X_2 = \mathcal{P}(x_2(t))$, and $x_1(t)$, $x_2(t)$ are sinusoids of the form (3.1).

The operators $\mathcal{P}$ and $\mathcal{P}^{-1}$ are *homogeneous*:

$$\begin{aligned} \mathcal{P}(Kx(t)) &= K\mathcal{P}(x(t)) \\ \mathcal{P}^{-1}(KX) &= K\mathcal{P}^{-1}(X) \end{aligned} \tag{3.9}$$

for any real constant K, with $X = \mathcal{P}(x(t))$.

Also, the two operators are *additive*:

$$\begin{aligned} \mathcal{P}(x_1(t) + x_2(t)) &= \mathcal{P}(x_1(t)) + \mathcal{P}(x_2(t)) \\ \mathcal{P}^{-1}(X_1 + X_2) &= \mathcal{P}^{-1}(X_1) + \mathcal{P}^{-1}(X_2) \end{aligned} \tag{3.10}$$

The phasor transform, $\mathcal{P}$, maps the operation of differentiating, D, into the multiplication by $j\omega$:

$$\begin{aligned} \mathcal{P}(\mathrm{D}x(t)) &= j\omega\mathcal{P}(x(t)) = j\omega X \\ \mathcal{P}^{-1}(j\omega X) &= \mathrm{D}\mathcal{P}^{-1}(X) = \mathrm{D}x(t) \end{aligned} \tag{3.11}$$

where $X = \mathcal{P}(x(t))$, and D denotes differentiation with respect to time, $\mathrm{D}x(t) = \mathrm{d}x(t)/\mathrm{d}t$.

3.1.3 Application of the Phasor Transform

The phasor representation of sinusoids is used mainly in the computation of the sinusoidal particular solution of ordinary linear differential equations with real constant coefficients when the forcing function is sinusoid; this computation is referred to as the *phasor method*.

Consider a lumped LTI system with a single input $x(t)$ and a single output $y(t)$, described by the following differential equation:

$$\begin{aligned} &a_n\mathrm{D}^n y + a_{n-1}\mathrm{D}^{n-1}y + \cdots + a_1\mathrm{D}y + a_0 y \\ &= b_m\mathrm{D}^m x + b_{m-1}\mathrm{D}^{m-1}x + \cdots + b_1\mathrm{D}x + b_0 x \end{aligned} \tag{3.12}$$

that is,

$$\sum_{i=0}^{n} a_i\mathrm{D}^i y = \sum_{k=0}^{m} b_k\mathrm{D}^k x \tag{3.13}$$

or, by introducing the polynomial differential operators,

$$A(\mathrm{D})\,y = B(\mathrm{D})\,x \tag{3.14}$$

where $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ are real numbers. If the input is a sinusoid given by (3.1), then we assume that a **sinusoidal** particular solution of Eq. (3.12) is of the form

$$y_{\mathrm{p}}(t) = Y_{\mathrm{m}}\cos(\omega t + \eta) \tag{3.15}$$

We apply the phasor transform (3.4) to both sides of Eq. (3.13) to obtain

$$\mathcal{P}\left(\sum_{i=0}^{n} a_i\mathrm{D}^i y_{\mathrm{p}}\right) = \mathcal{P}\left(\sum_{k=0}^{m} b_k\mathrm{D}^k x\right)$$

From the additivity (3.10) and homogeneity (3.9) property this becomes

$$\sum_{i=0}^{n} a_i \mathcal{P}\big(\mathrm{D}^i y_\mathrm{p}\big) = \sum_{k=0}^{m} b_k \mathcal{P}\big(\mathrm{D}^k x\big)$$

and from the differentiating property (3.11), applied repeatedly, we get

$$\sum_{i=0}^{n} a_i\,(j\omega)^i Y = \sum_{k=0}^{m} b_k\,(j\omega)^k X \tag{3.16}$$

with $Y = \mathcal{P}(y_\mathrm{p}(t))$, or, equivalently,

$$\left(\sum_{i=0}^{n} a_i\,(j\omega)^i\right) Y = \left(\sum_{k=0}^{m} b_k\,(j\omega)^k\right) X$$

By using the compact notation similar to Eq. (3.14) we may write

$$A(j\omega)\,Y = B(j\omega)X \tag{3.17}$$

Thus, if

$$A(j\omega) \neq 0 \tag{3.18}$$

we compute the phasor of the sinusoidal particular solution

$$Y = \frac{B(j\omega)}{A(j\omega)} X \tag{3.19}$$

where

$$A(j\omega) = \sum_{i=0}^{n} a_i\,(j\omega)^i \tag{3.20}$$

$$B(j\omega) = \sum_{k=0}^{m} b_k\,(j\omega)^k \tag{3.21}$$

Equation (3.16) can be obtained directly from Eq. (3.13) by replacing the ith derivatives of $y(t)$ with $(j\omega)^i Y$, for $i = 0$ to n, and by replacing the kth derivatives of $x(t)$ with $(j\omega)^k X$, for $k = 0$ to m. Also, we can write Eq. (3.17) directly from Eq. (3.14) by replacing the differential operator D with $j\omega$, $x(t)$ with X, and $y(t)$ with Y. Note that polynomial differential operators $A(\mathrm{D})$ and $B(\mathrm{D})$ are mapped into algebraic polynomials in complex variable $j\omega$ [Eqs. (3.20) and (3.21)].

The sinusoidal particular solution is found by the inverse phasor transform (3.5)

$$y_\mathrm{p}(t) = \mathcal{P}^{-1}(Y) = \mathcal{P}^{-1}(Y_\mathrm{m}\,\mathrm{e}^{j\eta}) = Y_\mathrm{m}\cos(\omega t + \eta) \tag{3.22}$$

where

$$Y_\mathrm{m} = \left|\frac{B(j\omega)}{A(j\omega)} X\right| = \left|\frac{B(j\omega)}{A(j\omega)}\right| X_\mathrm{m} \tag{3.23}$$

and

$$\eta = \arg\left(\frac{B(j\omega)}{A(j\omega)}X\right) = \arg\left(\frac{B(j\omega)}{A(j\omega)}\,e^{j\xi}\right) \tag{3.24}$$

Notice that the condition (3.18) must be satisfied for the value of ω under consideration. The sinusoidal particular solution (3.22) does not exist if this condition is not met—that is, if ω is such that $A(j\omega) = 0$.

The complex quantity $j\omega$ is often referred to as the *complex frequency of the excitation*. We shall frequently designate the complex frequency by s.

3.1.4 Sinusoidal Steady-State Response

Let us consider an LTI system characterized by Eq. (3.12) and driven by a single causal sinusoidal excitation:

$$x(t) = X_{\mathrm{m}} \cos(\omega t + \xi)\, u(t) = x_{\mathrm{s}}(t) u(t) \tag{3.25}$$

where $u(t)$ stands for the unit step function, and $x_{\mathrm{s}}(t)$ represents a sinusoid. We shall assume that the initial conditions are defined at time $t = t_0^- = 0^-$,

$$t_0^- = \lim_{\substack{t \to t_0 \\ t < t_0}} t$$

and the complete response of the system will be analyzed for $t \geq t_0$. Our goal is to examine what will response become as time approaches infinity.

The *complete response* of a system is the response of the system to both an input and the initial conditions. The complete response, $y(t)$, is the sum of the zero-input response, $y_0(t)$, and the zero-state response, $y_x(t)$:

$$y(t) = y_0(t) + y_x(t)$$

First, let us examine the *zero-input response* that is defined as a response of a system with no applied input. We have to solve the homogeneous differential equation

$$A(\mathrm{D})\, y_0 = 0 \tag{3.26}$$

If the corresponding characteristic equation

$$A(s) = \sum_{i=0}^{n} a_i s^i = 0 \tag{3.27}$$

has n distinct roots $s_1, s_2, \ldots, s_n$, any solution of (3.26), for $t \geq t_0$, may be written in the form

$$y_0(t) = \sum_{i=1}^{n} K_i\, e^{s_i t} \tag{3.28}$$

where the constants $K_1, K_2, \ldots, K_n$ are appropriately chosen so that the solution satisfies the prescribed initial conditions

$$\begin{aligned} y_0(t_0^-) &= y(t_0^-) \\ \mathrm{D}y_0(t_0^-) &= \mathrm{D}y(t_0^-) \\ &\vdots \\ \mathrm{D}^{n-1}y_0(t_0^-) &= \mathrm{D}^{n-1}y(t_0^-) \end{aligned} \tag{3.29}$$

Here, we use the notation

$$\begin{aligned} y_0(t_0^-) &= \lim_{\substack{t\to t_0 \\ t<t_0}} y_0(t) \\ \mathrm{D}y_0(t_0^-) &= \lim_{\substack{t\to t_0 \\ t<t_0}} \frac{\mathrm{d}y_0(t)}{\mathrm{d}t} \\ &\vdots \\ \mathrm{D}^k y_0(t_0^-) &= \lim_{\substack{t\to t_0 \\ t<t_0}} \frac{\mathrm{d}^k y_0(t)}{\mathrm{d}t^k} \end{aligned}$$

If the characteristic equation (3.27) has ν distinct roots $s_1, s_2, \ldots, s_\nu$, where $\nu < n$, and if the multiplicities of these roots are $k_1, k_2, \ldots, k_\nu$ respectively, then

$$k_1 + k_2 + \cdots + k_\nu = n$$

and any solution of (3.26) can be written as

$$y_0(t) = \sum_{i=1}^{\nu} p_i(t)\, \mathrm{e}^{s_i t} \tag{3.30}$$

where $p_1(t), p_2(t), \ldots, p_\nu(t)$ are polynomials in the variable t of degree $k_1 - 1, k_2 - 1, \ldots, k_\nu - 1$, respectively. The coefficients of these polynomials must be chosen so that the solution satisfies the initial conditions (3.29).

The roots of the characteristic equation (3.27), $s_1, s_2, \ldots,$ are referred to as the *natural frequencies* of the system. If the coefficients of the characteristic equations are real, the natural frequencies are real or occur in complex conjugate pairs. We shall be concerned with systems that are characterized by equations with real coefficients.

Equations (3.28) and (3.30) indicate that irrespective of the initial state and provided that all natural frequencies are in the open left-half plane, the zero-input response tends to **zero** as $t \to \infty$. The *open left-half plane* consists of the left half of the complex plane with the imaginary axis excluded. In other words, the open left-half plane includes all points with negative real parts.

When an LTI system has all its natural frequencies in the open left-half plane, we say that the system is *asymptotically stable*. Any zero-input response of an asymptotically stable system vanishes (approaches zero) as $t \to \infty$. Real systems, especially filters, are designed to be stable. If one or more natural frequencies are in the open right-half plane, we say that the system is *unstable*. For most initial states, the zero-input response of an unstable system becomes infinite as $t \to \infty$. Unstable systems are not within the scope of this book.

If the system has only one complex conjugate pair of purely imaginary simple natural frequencies, say $j\omega_0$ and $-j\omega_0$, and all other natural frequencies are in the open left-half plane, the zero-input response contains an oscillatory part at the angular frequency ω_0, and it becomes sinusoidal as $t \to \infty$. This response is called the *zero-input sinusoidal steady-state* response.

If the system has several imaginary natural frequencies that are simple, $\pm j\omega_{01}$, $\pm j\omega_{02}, \ldots$, and all other natural frequencies are in the open left-half plane, the zero-input response will tend to the sum of sinusoids with angular frequencies $\omega_{01}, \omega_{02}, \ldots$, as $t \to \infty$. We shall call this response the *zero-input steady-state* response; it is periodic but it is not sinusoidal.

The *zero-state response* of an LTI system, $y_x(t)$, is the response of the system to an input applied at some time t_0 subject to the condition that the system be in zero state just prior to the application of the input (that is, at time t_0^-). In calculating zero-state responses, our primary interest is the behavior of the response for $t \geq t_0$. For this reason we adopt the convention that the input and the zero-state response are taken to be identically zero for $t < t_0$. Unless otherwise stated, we assume that $t_0 = 0$. We have to solve the equation

$$\sum_{i=0}^{n} a_i \mathrm{D}^i y_x = \sum_{k=0}^{m} b_k \mathrm{D}^k x \tag{3.31}$$

and we shall assume that $n \geq m$. By definition, the initial conditions are zero and

$$\begin{aligned} y_x(t_0^-) &= 0 \\ \mathrm{D}y_x(t_0^-) &= 0 \\ &\vdots \\ \mathrm{D}^{n-1}y_x(t_0^-) &= 0 \end{aligned} \tag{3.32}$$

The causal input $x(t)$ is the product of an elementary function, $x_s(t)$, and the unit step function $u(t)$. *Elementary functions* are polynomials, exponential functions, trigonometric functions, and so on, and combination of these functions; elementary functions do not contain step functions and impulse functions.

Under the condition $n \geq m$ the response $y_x(t)$ must be of the form

$$y_x(t) = y_{xs}(t)u(t) \tag{3.33}$$

where $y_{xs}(t)$ represents an elementary function and Eq. (3.31) becomes

$$\sum_{i=0}^{n} a_i \mathrm{D}^i (y_{xs}(t)u(t)) = \sum_{k=0}^{m} b_k \mathrm{D}^k \left(x_s(t)u(t)\right) \tag{3.34}$$

Next, we apply rules for differentiating the product of an elementary function and the step function

$$\mathrm{D}(x_\mathrm{s}(t)u(t)) = (\mathrm{D}x_\mathrm{s}(t))\,u(t) + \big(x_\mathrm{s}(0^+)\big)\mathrm{D}u(t)$$

$$\mathrm{D}^2(x_\mathrm{s}(t)u(t)) = \big(\mathrm{D}^2x_\mathrm{s}(t)\big)u(t) + \big(\mathrm{D}x_\mathrm{s}(0^+)\big)\mathrm{D}u(t) + \big(x_\mathrm{s}(0^+)\big)\mathrm{D}^2u(t)$$

$$\mathrm{D}^k(x_\mathrm{s}(t)u(t)) = \big(\mathrm{D}^kx_\mathrm{s}(t)\big)\,u(t) + \sum_{q=1}^{k}\big(\mathrm{D}^{k-q}x_\mathrm{s}(0^+)\big)\mathrm{D}^q u(t) \tag{3.35}$$

where

$$x_\mathrm{s}(0^+) = \lim_{\substack{t\to 0\\ t>0}} x_\mathrm{s}(t)$$

$$\mathrm{D}^k x_\mathrm{s}(t) = \frac{\mathrm{d}^k x_\mathrm{s}(t)}{\mathrm{d}t^k}$$

$$\mathrm{D}^0 x_\mathrm{s}(t) = x_\mathrm{s}(t)$$

$$\mathrm{D}^k x_\mathrm{s}(0^+) = \lim_{\substack{t\to 0\\ t>0}} \frac{\mathrm{d}^k x_\mathrm{s}(t)}{\mathrm{d}t^k}$$

Equation (3.34) transforms into

$$\sum_{i=0}^{n} a_i\left(\big(\mathrm{D}^i y_{xs}(t)\big)\,u(t) + \sum_{q=1}^{i}\big(\mathrm{D}^{i-q} y_{xs}(0^+)\big)\,\mathrm{D}^q u(t)\right)$$

$$= \sum_{k=0}^{m} b_k\left(\big(\mathrm{D}^k x_\mathrm{s}(t)\big)\,u(t) + \sum_{q=1}^{k}\big(\mathrm{D}^{k-q} x_\mathrm{s}(0^+)\big)\,\mathrm{D}^q u(t)\right)$$

$$\sum_{i=0}^{n} a_i\,\big(\mathrm{D}^i y_{xs}(t)\big)\,u(t) + \sum_{i=1}^{n} a_i \sum_{q=1}^{i}\big(\mathrm{D}^{i-q} y_{xs}(0^+)\big)\,\mathrm{D}^q u(t)$$

$$= \sum_{k=0}^{m} b_k\,\big(\mathrm{D}^k x_\mathrm{s}(t)\big)\,u(t) + \sum_{k=1}^{m} b_k \sum_{q=1}^{k}\big(\mathrm{D}^{k-q} x_\mathrm{s}(0^+)\big)\,\mathrm{D}^q u(t)$$

which yields

$$\sum_{i=0}^{n} a_i \mathrm{D}^i y_{xs}(t) = \sum_{k=0}^{m} b_k \mathrm{D}^k x_\mathrm{s}(t) \tag{3.36}$$

or

$$A(\mathrm{D})\,y_{xs} = B(\mathrm{D})\,x_\mathrm{s} \tag{3.37}$$

and

$$\sum_{i=1}^{n} a_i \sum_{q=1}^{i} \left(\mathrm{D}^{i-q} y_{xs}(0^+)\right) \mathrm{D}^q u(t) \;=\; \sum_{k=1}^{m} b_k \sum_{q=1}^{k} \left(\mathrm{D}^{k-q} x_{\mathrm{s}}(0^+)\right) \mathrm{D}^q u(t) \tag{3.38}$$

Equating the like terms in (3.38), we formulate a system of linear equations from which we determine the initial conditions $y_{xs}(0^+)$, $\mathrm{D}y_{xs}(0^+)$, $\mathrm{D}^2 y_{xs}(0^+)$, $\ldots$, $\mathrm{D}^{n-1} y_{xs}(0^+)$. We use these initial conditions to solve Eq. (3.36)

$$\begin{aligned}
\mathrm{D}u(t) \;&\leftrightarrow\; \sum_{i=1}^{n} a_i \mathrm{D}^{i-1} y_{xs}(0^+) \;=\; \sum_{k=1}^{m} b_k \mathrm{D}^{k-1} x_{\mathrm{s}}(0^+) \\
\mathrm{D}^2 u(t) \;&\leftrightarrow\; \sum_{i=2}^{n} a_i \mathrm{D}^{i-2} y_{xs}(0^+) \;=\; \sum_{k=2}^{m} b_k \mathrm{D}^{k-2} x_{\mathrm{s}}(0^+) \\
&\vdots
\end{aligned} \tag{3.39}$$

If $n > m$ the response $y_x(t)$ is continuous at the instant $t = t_0 = 0$ and

$$y_x(0^+) \;=\; y_{xs}(0^+) \;=\; y_x(0^-) \;=\; 0 \tag{3.40}$$

We say that the initial conditions are *consistent*. For $n = m$, Eq. (3.40) does not hold, the response $y_x(t)$ has a jump at t_0, and the initial conditions are *inconsistent*. The set of equations (3.39) must be solved to find the required initial conditions at $t = t_0^+ = 0^+$.

The general solution of Eq. (3.36) is the sum of two terms:

$$y_{xs}(t) \;=\; y_{\mathrm{xsh}}(t) \;+\; y_{\mathrm{xsp}}(t) \tag{3.41}$$

where $y_{\mathrm{xsh}}(t)$ is the solution of the homogeneous equation

$$A(\mathrm{D})\, y_{\mathrm{xsh}} \;=\; 0 \tag{3.42}$$

and $y_{\mathrm{xsp}}(t)$ is a particular solution.

The term $y_{\mathrm{xsh}}(t)$ is of the form (3.30):

$$y_{\mathrm{xsh}}(t) \;=\; \sum_{i=1}^{\nu} P_i(t)\, \mathrm{e}^{s_i t}$$

where $P_1(t), P_2(t), \ldots, P_\nu(t)$ are polynomials in the variable t of degree $k_1 - 1, k_2 - 1, \ldots, k_\nu - 1$, respectively; $k_1, k_2, \ldots, k_\nu$ are multiplicities of the natural frequencies.

The particular solution is sinusoidal

$$y_{\mathrm{xsp}}(t) \;=\; K_{\mathrm{c}} \cos(\omega t) \;+\; K_{\mathrm{s}} \sin(\omega t) \;=\; Y_{\mathrm{m}} \cos(\omega t \;+\; \eta) \tag{3.43}$$

if the complex frequency $j\omega$ is not a root of the characteristic equation; K_{c} and K_{s} are real constants. If $j\omega$ is a natural frequency of order λ, then

$$y_{\mathrm{xsp}}(t) \;=\; p_{\mathrm{c}}(t) \cos(\omega t) \;+\; p_{\mathrm{s}}(t) \sin(\omega t) \tag{3.44}$$

where $p_{\mathrm{c}}(t)$ and $p_{\mathrm{s}}(t)$ are polynomials in t of degrees λ.

The complete response is

$$y(t) = y_0(t) + y_{\mathrm{xsh}}(t)u(t) + y_{\mathrm{xsp}}(t)u(t), \qquad t \geq t_0 \tag{3.45}$$

or generally

$$y(t) = \sum_{i=1}^{\nu} p_i(t)\,\mathrm{e}^{s_i t} + \left(\sum_{i=1}^{\nu} P_i(t)\,\mathrm{e}^{s_i t}\right)u(t) + (p_\mathrm{c}(t)\cos(\omega t) + p_\mathrm{s}(t)\sin(\omega t))u(t)$$

and the above analysis allows us to state the following facts [3]:

- Irrespective of the initial state and provided that all the natural frequencies are in the open left-half plane, the complete response of an LTI system driven by a sinusoidal input will become sinusoidal as $t \to \infty$. This sinusoidal response is called the *sinusoidal steady-state response*.
- The sinusoidal steady-state response always has the same frequency as the input.
- The sinusoidal steady-state response can be obtained efficiently by the phasor method.

The systems that we analyze in this book have natural frequencies in the open left-half plane; therefore, the sinusoidal steady-state response exists for these systems at any angular frequency of the sinusoidal input.

3.1.5 Nonsinusoidal Steady-State Response

Practical systems are often driven by several excitations. A system variable, the response $y(t)$, can be found from a differential equation of the form

$$A(\mathrm{D})\, y = \sum_{g=1}^{N} B_g(\mathrm{D})x_g \tag{3.46}$$

The inputs to the system, $x_g(t)$, are supposed to be sinusoidal

$$x_g(t) = X_{\mathrm{m}g}\cos(\omega_g t + \xi_g)\,u(t) = x_{\mathrm{s}g}(t)\,u(t) \tag{3.47}$$

with arbitrary and different angular frequencies $\omega_1, \omega_2, \ldots, \omega_N$. The degree of $A(\mathrm{D})$ is assumed not to be less than the degree of $B_g(\mathrm{D})$.

We focus on lumped LTI systems with natural frequencies in the open left-half plane.

According to the principal of superposition the complete response, $y(t)$, is the sum of the zero-input response, $y_0(t)$, and the zero-state responses that would exist if each input sinusoid were acting alone on the system:

$$y(t) = y_0(t) + \sum_{g=1}^{N} y_{xg}(t) = y_0(t) + \sum_{g=1}^{N} y_{\mathrm{xh}g}(t) + \sum_{g=1}^{N} y_{\mathrm{xp}g}(t), \qquad t \geq 0 \tag{3.48}$$

where $y_{\mathrm{xh}g}(t)$ is the solution of the homogeneous equation, and $y_{\mathrm{xp}g}(t)$ is the particular solution of the form

$$y_{\mathrm{xp}g}(t) = Y_{\mathrm{m}g}\cos(\omega_g t + \eta_g)u(t) \tag{3.49}$$

for $g = 1, 2, \ldots, N$.

Whatever the initial conditions may be, as $t \to \infty$, the complete response $y(t)$ becomes arbitrarily close to the value of its particular solution given by

$$y_\mathrm{p}(t) = \sum_{g=1}^{N} Y_{\mathrm{m}g} \cos(\omega_g t + \eta_g) u(t) \tag{3.50}$$

that is,

$$y(t) \to \sum_{g=1}^{N} Y_{\mathrm{m}g} \cos(\omega_g t + \eta_g), \qquad t \to \infty$$

The response $y(t)$, as $t \to \infty$, is called the *steady state* (not sinusoidal steady state).

The steady state resulting from several input sinusoids is the sum of the sinusoidal steady states that would exist if each input sinusoid were acting alone on the system.

We can use the phasor method to obtain $y_{\mathrm{xp}g}(t)$. First, we find the phasors of the input sinusoids (3.47):

$$X^{(\omega_g)} = \mathcal{P}_{\omega_g}(x_g(t)) = X_{\mathrm{m}g}\, \mathrm{e}^{j\xi_g}, \qquad g = 1, 2, \ldots, N \tag{3.51}$$

Next, we compute the output phasors as

$$Y^{(\omega_g)} = \frac{B_g(j\omega_g)}{A(j\omega_g)} X^{(\omega_g)}, \qquad g = 1, 2, \ldots, N \tag{3.52}$$

Finally, we determine the particular solution (i.e., steady state) by taking the inverse phasor transform of the output phasors:

$$y_\mathrm{p}(t) = u(t) \sum_{g=1}^{N} \mathcal{P}_{\omega_g}^{-1}(Y^{(\omega_g)}) \tag{3.53}$$

As we expect, the superposition holds in the steady state.

The *constant steady state* can be readily obtained as a special (trivial) case when

$$\omega_g = 0, \qquad \xi_g = 0$$
$$g = 1, 2, \ldots, N$$

that is, when all inputs are constant (time-invariant) signals.

3.1.6 Transfer Function and Frequency Response

The principal advantage of the phasor method is efficient computation of the sinusoidal steady-state response by solving an algebraic equation rather than a differential equation. Equation (3.19) shows that a linear relation exists between the output phasor and the input phasor. For a single-input lumped LTI system in sinusoidal steady state we can represent the input, output, and other system variables with phasors. Our target is to study complex functions that relate phasors in an LTI system.

We define a *transfer function* (also called the *frequency response*), for a single-input lumped LTI system in the sinusoidal steady state, to be the ratio of the output phasor to the input phasor

$$H(j\omega) = \frac{Y}{X} = \frac{B(j\omega)}{A(j\omega)} \tag{3.54}$$

The transfer function depends on system parameters and the angular frequency ω. Generally, $H(j\omega)$ is a rational function of the complex frequency $j\omega$, and it contains all the needed information concerning the sinusoidal steady-state response.

The magnitude of the transfer function

$$M(\omega) = |H(j\omega)| = \left|\frac{B(j\omega)}{A(j\omega)}\right| \tag{3.55}$$

and its phase

$$\Phi(\omega) = \arg(H(j\omega)) = \arg\left(\frac{B(j\omega)}{A(j\omega)}\right) \tag{3.56}$$

are called the *frequency response*; $M(\omega)$ is the *magnitude response*, and $\Phi(\omega)$ is the *phase response*. The curves of $M(\omega)$ and $\Phi(\omega)$ versus ω or $\log(\omega)$ are called the *frequency characteristics* for a specified input and output. The amplitude response is often expressed in *decibels* (dB) as follows:

$$M_{\mathrm{dB}}(\omega) = 20\log_{10}(M(\omega)) \tag{3.57}$$

The quantity $M_{\mathrm{dB}}(\omega)$ is usually called the *gain*. The negative of the gain is called the *attenuation* or *loss*. The plot of the attenuation versus ω is the *attenuation characteristic* or *loss characteristic*. Given the phase response $\Phi(\omega)$, the *group delay* is defined as

$$\tau(\omega) = -\frac{\mathrm{d}\Phi(\omega)}{\mathrm{d}\omega} \tag{3.58}$$

and the corresponding plot against ω is known as *delay characteristic*. The frequency characteristics can also be plotted versus f or $\log(f)$.

Given the transfer function $H(j\omega) = M(\omega)\,\mathrm{e}^{j\Phi(\omega)}$ and the sinusoidal input $x(t) = X_{\mathrm{m}}\cos(\omega t + \xi)$, the sinusoidal steady-state response is $y(t) = M(\omega)X_{\mathrm{m}}\cos(\omega t + \xi + \Phi(\omega))$. In other words, the amplitude of the output is obtained by taking the product of the transfer function magnitude and the amplitude of the input; the phase of the output is obtained by adding the phase of the transfer function and the phase of the input.

If we denote the complex frequency $j\omega$ by the symbol s, we can formally write

$$H(j\omega) = H(s) = \frac{B(s)}{A(s)}, \qquad s = j\omega \tag{3.59}$$

The roots of the transfer function denominator $A(s)$ are called the *poles* of the transfer function. In fact, the poles are the natural frequencies of the system. We compute the poles from the equation

$$A(s) = \text{denominator}(H(s)) = 0$$

and designate these poles by $s_{\mathrm{p}1}, s_{\mathrm{p}2}, s_{\mathrm{p}3}, \ldots, s_{\mathrm{p}n}$.

In a similar way, the roots of the transfer function numerator $B(s)$ are called the *zeros* of the transfer function. We obtain zeros from the equation

$$B(s) = \text{numerator}(H(s)) = 0$$

and designate these zeros by $s_{z1}, s_{z2}, s_{z3}, \ldots, s_{zm}$.

The transfer function can be written as a function of $s = j\omega$ in the factored form as

$$H(s) = H_0 \frac{\prod_{k=1}^{m}(s - s_{zk})}{\prod_{i=1}^{n}(s - s_{pi})}, \qquad s = j\omega \tag{3.60}$$

The form (3.60) is known as the *pole-zero representation* of the transfer function. The real constant H_0 is called the *scale factor*.

A transfer function with no zeros is called the *all-pole transfer function* and is of the form

$$H(s) = H_0 \frac{1}{\prod_{i=1}^{n}(s - s_{pi})}, \qquad s = j\omega \tag{3.61}$$

Since the polynomials $A(s)$ and $B(s)$ have real coefficients, zeros and poles must be real or occur in complex conjugate pairs.

For each complex pole $s_{pi} = \sigma_{pi} + j\omega_{pi}$ we define the *pole magnitude*

$$p_i = |s_{pi}| = \sqrt{\sigma_{pi}^2 + \omega_{pi}^2} \tag{3.62}$$

and the pole *quality factor* (also called the Q-factor)

$$Q_{pi} = \frac{|s_{pi}|}{-2\sigma_{pi}} = \frac{p_i}{-2\sigma_{pi}} \tag{3.63}$$

For the systems that we study, the poles are in the open left-half plane and the pole Q-factor is a positive number. The minimal value of Q-factor is $1/2$ and occurs for poles with vanishingly small imaginary parts. Purely imaginary poles have infinite quality factors. In the same way we define the magnitude and the Q-factor of the transfer-function zeros.

The simplest transfer function is the first-order transfer function

$$H(s) = \frac{b_1 s + b_0}{a_1 s + a_0}, \qquad s = j\omega, \quad a_1 \neq 0 \tag{3.64}$$

Systems described by (3.64) we call *first-order sections*.

Transfer functions of the form

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}, \qquad s = j\omega, \quad a_2 \neq 0 \tag{3.65}$$

are called *second-order transfer functions*, and they play an important role in analysis and design of LTI systems. A system characterized by (3.65) we call a *biquadratic section* or *biquad*.

Any transfer function (3.54) can be expressed as a product of first-order (3.64) and second-order (3.65) transfer functions, which implies that any LTI system can be resolved into first-order sections and biquads.

Partial Transfer Function and Transfer-Function Matrix. Consider a multiple-input multiple-output system with N sinusoidal inputs, $x_1, x_2, \ldots, x_k, \ldots, x_N$, with equal angular frequencies $\omega_1 = \omega_2 = \cdots = \omega_N = \omega$, and assume that the system has L outputs $y_1, y_2, \ldots, y_i, \ldots, y_L$. We define the *partial transfer function* between the ith output and the kth input to be the ratio of the output phasor Y_i to the input phasor X_k, with the other inputs being identically zero:

$$H_{ik}(j\omega) = \left.\frac{Y_i}{X_k}\right|_{x_1=\cdots=x_{k-1}=x_{k+1}=\cdots=x_N=0} \tag{3.66}$$

The input–output description of the system can be extended to phasors and expressed in a matrix form

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_i \\ \vdots \\ Y_L \end{bmatrix} = \begin{bmatrix} H_{11}(j\omega) & H_{12}(j\omega) & \cdots & H_{1k}(j\omega) & \cdots & H_{1N}(j\omega) \\ H_{21}(j\omega) & H_{22}(j\omega) & \cdots & H_{2k}(j\omega) & \cdots & H_{2N}(j\omega) \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ H_{i1}(j\omega) & H_{i2}(j\omega) & \cdots & H_{ik}(j\omega) & \cdots & H_{iN}(j\omega) \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ H_{L1}(j\omega) & H_{L2}(j\omega) & \cdots & H_{Lk}(j\omega) & \cdots & H_{LN}(j\omega) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \\ \vdots \\ X_N \end{bmatrix}$$

where the matrix

$$\mathbf{H}(j\omega) = \begin{bmatrix} H_{11}(j\omega) & H_{12}(j\omega) & \cdots & H_{1k}(j\omega) & \cdots & H_{1N}(j\omega) \\ H_{21}(j\omega) & H_{22}(j\omega) & \cdots & H_{2k}(j\omega) & \cdots & H_{2N}(j\omega) \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ H_{i1}(j\omega) & H_{i2}(j\omega) & \cdots & H_{ik}(j\omega) & \cdots & H_{iN}(j\omega) \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ H_{L1}(j\omega) & H_{L2}(j\omega) & \cdots & H_{Lk}(j\omega) & \cdots & H_{LN}(j\omega) \end{bmatrix}$$

is called the *transfer-function matrix* of the system. Whenever a transfer-function matrix $\mathbf{H}(j\omega)$ is used, the inputs are assumed to have equal angular frequencies ω. The principle of superposition holds for phasors if all inputs have the same angular frequency ω.

3.1.7 Application Example of Phasor Method

The computational efficiency of the phasor method is best demonstrated by illustrative application examples. We shall solve a problem that is important in analysis and design of LTI systems.

Solving a Second-Order System

Problem 3.1.1

A system is characterized by the equation

$$a_2 \mathrm{D}^2 y + a_1 \mathrm{D} y + a_0 y = b_2 \mathrm{D}^2 x + b_1 \mathrm{D} x + b_0 x$$

with real coefficients such that

$$a_2 > 0$$
$$a_1 > 0$$
$$a_0 > 0$$
$$a_1^2 < a_2 a_0$$
$$b_2 \neq 0$$

The input is a causal sinusoid:

$$x(t) = X_{\mathrm{m}} \cos(\omega t) u(t)$$

The initial conditions are given at $t = t_0 = 0$:

$$y(0^-) = U$$
$$\mathrm{D} y(0^-) = V$$

Find the complete response for $t \geq 0$, and check whether the system can reach the sinusoidal steady state (numerical values: $a_2 = 1$, $a_1 = 2$, $a_0 = 5$, $b_2 = 3$, $b_1 = 4$, $b_0 = 6$, $U = 2$, $V = 6$, $\omega = 1$, $X_{\mathrm{m}} = 1$).

Solution

The characteristic equation is

$$A(s) = a_2 s^2 + a_1 s + a_0 = 0$$

and the natural frequencies are

$$s_1 = \sigma_1 + j\omega_1 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} = \frac{-a_1}{2a_2} + j\frac{-\sqrt{4a_2 a_0 - a_1^2}}{2a_2} = -1 - j2$$

$$s_2 = \sigma_2 + \mathrm{j}\omega_2 = \sigma_1 - j\omega_1 = s_1^* = -1 + j2$$

with

$$\sigma_1 = \frac{-\,a_1}{2a_2} = -1 < 0, \quad \omega_1 = \frac{-\,\sqrt{4a_2 a_0 - a_1^2}}{2a_2} = -2$$

The natural frequencies are different, are simple, occur in a complex conjugate pair, and are in the open left-half complex plane.

The zero-input response, $y_0(t)$, is the solution of the equation

$$a_2 \mathrm{D}^2 y_0 + a_1 \mathrm{D} y_0 + a_0 y_0 = 0$$

and it is of the form

$$y_0(t) = K_1\,\mathrm{e}^{s_1 t} + K_2\,\mathrm{e}^{s_2 t}$$

The real constants K_1 and K_2 are chosen to meet the initial conditions:

$$y_0(0) = y(0^-) = U$$

$$\mathrm{D}y_0(0) = \mathrm{D}y(0^-) = V$$

It follows that

$$K_1 + K_2 = U$$

$$s_1 K_1 + s_2 K_2 = V$$

$$K_1 = \frac{s_2 U - V}{s_2 - s_1} = \frac{U}{2} + j\frac{\sigma_1 U - V}{2\omega_1} = 1 + j2$$

$$K_2 = \frac{s_1 U - V}{s_1 - s_2} = \frac{U}{2} - j\frac{\sigma_1 U - V}{2\omega_1} = K_1^* = 1 - j2$$

$$y_0(t) = \mathrm{e}^{\sigma_1 t}\left(U\cos(\omega_1 t) + \frac{V - \sigma_1 U}{\omega_1}\sin(\omega t)\right), \qquad t \geq 0$$

$$y_0(t) = \mathrm{e}^{-t}\,(2\cos(2t) + 4\sin(2t)), \qquad t \geq 0$$

The zero-state response $y_x(t)$ is of the form

$$y_x(t) = y_{\mathrm{xs}}(t)\,u(t)$$

The differential equation that characterizes the system becomes

$$\begin{aligned} &a_2 \mathrm{D}^2(y_{\mathrm{xs}}(t)u(t)) + a_1 \mathrm{D}(y_{\mathrm{xs}}(t)u(t)) + a_0(y_{\mathrm{xs}}(t)u(t)) \\ &\quad = b_2 \mathrm{D}^2(x_{\mathrm{s}}(t)u(t)) + b_1 \mathrm{D}(x_{\mathrm{s}}(t)u(t)) + b_0(x_{\mathrm{s}}(t)u(t)) \end{aligned}$$

where $x_{\mathrm{s}}(t) = X_{\mathrm{m}}\cos(\omega t)$. It expands to

$$\begin{aligned} &a_2\big(\mathrm{D}^2 y_{\mathrm{xs}}(t)\big)\,u(t) + a_2\big(\mathrm{D}y_{\mathrm{xs}}(0^+)\big)\,\mathrm{D}u(t) + a_2\big(y_{\mathrm{xs}}(0^+)\big)\mathrm{D}^2 u(t) \\ &\quad + a_1(\mathrm{D}y_{\mathrm{xs}}(t))\,u(t) + a_1\big(y_{\mathrm{xs}}(0^+)\big)\,\mathrm{D}u(t) \\ &\quad + a_0 y_{\mathrm{xs}}(t)\,u(t) \\ &= b_2\big(\mathrm{D}^2 x_{\mathrm{s}}(t)\big)u(t) + b_2\big(\mathrm{D}x_{\mathrm{s}}(0^+)\big)\mathrm{D}u(t) + b_2\big(x_{\mathrm{s}}(0^+)\big)\mathrm{D}^2 u(t) \\ &\quad + b_1(\mathrm{D}x_{\mathrm{s}}(t))\,u(t) + b_1\,\big(x_{\mathrm{s}}(0^+)\big)\,\mathrm{D}u(t) \\ &\quad + b_0 x_{\mathrm{s}}(t)u(t) \end{aligned}$$

with

$$x_{\mathrm{s}}(0^+) = X_{\mathrm{m}}\cos(\omega t)|_{t\to 0+} = X_{\mathrm{m}} = 1$$

$$\mathrm{D}x_{\mathrm{s}}(0^+) = -\omega X_{\mathrm{m}}\sin(\omega t)|_{t\to 0+} = 0$$

Equating the like terms we obtain the differential equation in $y_{xs}(t)$

$$a_2\mathrm{D}^2 y_{xs} + a_1\mathrm{D}y_{xs} + a_0 y_{xs} = b_2\mathrm{D}^2 x_s + b_1\mathrm{D}x_s + b_0 x_s$$

and the set of linear algebraic equations for the corresponding initial conditions that $y_{xs}(t)$ must meet

$$a_2\mathrm{D}y_{xs}(0^+) + a_1 y_{xs}(0^+) = b_2 \cdot 0 + b_1 X_m$$

$$a_2 y_{xs}(0^+) = b_2 X_m$$

The required initial conditions are

$$y_{xs}(0^+) = \frac{b_2}{a_2} X_m = 3$$

$$\mathrm{D}y_{xs}(0^+) = \frac{a_2 b_1 - a_1 b_2}{a_2^2} X_m = -2$$

and are inconsistent, i.e. $y_x(t)$ is discontinuous at $t = 0$.

The function $y_{xs}(t)$ is of the form

$$y_{xs}(t) = y_{xsh}(t) + y_{xsp}(t)$$

$y_{xsh}(t)$ is a solution of the homogeneous differential equation

$$a_2\mathrm{D}^2 y_{xsh} + a_1\mathrm{D}y_{xsh} + a_0 y_{xsh} = 0$$

and it takes the form

$$y_{xsh}(t) = C_1 \mathrm{e}^{s_1 t} + C_2 \mathrm{e}^{s_2 t}$$

The natural frequencies s_1 and s_2 are in the open left-half plane, thus, the particular solution $y_{xsp}(t)$ is sinusoidal

$$y_{xsp}(t) = Y_m \cos(\omega t + \eta)$$

and can be computed by the phasor method

$$y_{xsp}(t) = \mathcal{P}^{-1}\left(\frac{b_2(j\omega)^2 + b_1(j\omega) + b_0}{a_2(j\omega)^2 + a_1(j\omega) + a_0} X_m \mathrm{e}^{j0}\right)$$

$$Y_m = \left|\frac{b_2(j\omega)^2 + b_1(j\omega) + b_0}{a_2(j\omega)^2 + a_1(j\omega) + a_0} X_m\right| = \frac{2}{5}\sqrt{5}$$

$$\eta = \arg\left(\frac{b_2(j\omega)^2 + b_1(j\omega) + b_0}{a_2(j\omega)^2 + a_1(j\omega) + a_0} \mathrm{e}^{j0}\right) = -\operatorname{atan}\left(\frac{1}{2}\right)$$

Finally, we have

$$y_{xs}(t) = C_1 \mathrm{e}^{s_1 t} + C_2 \mathrm{e}^{s_2 t} + Y_m \cos(\omega t + \eta)$$

Constants C_1 and C_2 are chosen to satisfy the initial conditions $y_{xs}(0^+)$ and $\mathrm{D}y_{xs}(0^+)$:

$$y_{xs}(0^+) = C_1 + C_2 + Y_m \cos(\eta)$$

$$\mathrm{D}y_{xs}(0^+) = s_1 C_1 + s_2 C_2 - \omega Y_m \sin(\eta)$$

that is, the set of linear algebraic equations

$$\frac{b_2}{a_2} X_\mathrm{m} = C_1 + C_2 + Y_\mathrm{m}\cos(\eta)$$

$$\frac{a_2 b_1 - a_1 b_2}{a_2^2} X_\mathrm{m} = s_1 C_1 + s_2 C_2 - \omega Y_\mathrm{m}\sin(\eta)$$

uniquely determines $C_1 = \frac{11}{10} - j\frac{1}{20}$ and $C_2 = \frac{11}{10} + j\frac{1}{20}$.

The complete response $y(t)$, for $t \geq 0$, is as follows:

$$y(t) = \underbrace{K_1\,\mathrm{e}^{s_1 t} + K_2\,\mathrm{e}^{s_2 t}}_{y_0} + \underbrace{\left(C_1\,\mathrm{e}^{s_1 t} + C_2\,\mathrm{e}^{s_2 t}\right)}_{y_\mathrm{xsh}} u(t) + \underbrace{Y_\mathrm{m}\cos(\omega t + \eta)}_{y_\mathrm{xsp}}\, u(t)$$

where $K_1, K_2, C_1, C_2, Y_\mathrm{m}$, and η are known constants.

$$\begin{aligned} y(t) = {} & \mathrm{e}^{-t}(2\cos(2t) + 4\sin(2t)) \\ & + \mathrm{e}^{-t}(\frac{11}{5}\cos(2t) - \frac{1}{10}\sin(2t))\, u(t) \\ & + \frac{2}{5}\sqrt{5}\cos(t - \mathrm{atan}(\frac{1}{2}))\, u(t) \\ t \geq {} & 0 \end{aligned}$$

As $t \to \infty$ the exponential terms tend to zero because

$$\begin{aligned} \mathrm{Re}(s_1) &= \mathrm{Re}(s_2) < 0 \\ \lim_{t\to\infty} \mathrm{e}^{s_1 t} &= \lim_{t\to\infty} \left(\mathrm{e}^{\sigma_1 t}(\cos(\omega_1 t) + j\sin(\omega_1 t))\right) = 0, \qquad \sigma_1 < 0 \\ \lim_{t\to\infty} \mathrm{e}^{s_2 t} &= \lim_{t\to\infty} \mathrm{e}^{s_1^* t} = \left(\lim_{t\to\infty} \mathrm{e}^{s_1 t}\right)^* = 0 \end{aligned}$$

Therefore, as time tends to infinity the complete response approaches the particular solution.

The sinusoidal steady-state response is $Y_\mathrm{m}\cos(\omega t + \eta)$ for $t \gg t_0 = 0$.

3.2 FOURIER SERIES AND HARMONIC ANALYSIS

The theory of phasors and the "$j\omega$" method was entirely founded on the assumption of sinusoidal variation of signals. Nonsinusoidal periodic signals play an important part in electronics, telecommunications, control engineering, power engineering, and signal processing in general. The theory of nonsinusoidal periodic signals is based upon resolving them into sinusoidal components. Next, according to the principle of superposition, we can find the steady-state response of an LTI system to an arbitrary periodic input by applying the phasor method to each harmonic component.

3.2.1 The Fourier Series

Consider a real signal $x(t)$ of real variable t that satisfies the following conditions:

- $x(t+T) = x(t)$; that is, the signal is periodic having a period T.
- $x(t)$ is defined in the interval $\tau < t < \tau + T$.
- $x(t)$ and $\mathrm{d}x(t)/\mathrm{d}t$ are sectionally continuous in $\tau < t < \tau + T$.

A signal is called *sectionally continuous* or piecewise continuous in an interval if the interval can be subdivided into finite number of intervals in each of which the signal is continuous and has finite right- and left-hand limits. Then, at every point of continuity the periodic signal $x(t)$ can be represented by a series of the form

$$x(t) = C_0 + \sum_{n=1}^{+\infty} (A_n \cos(n\omega_1 t) + B_n \sin(n\omega_1 t)) \tag{3.67}$$

where

$$\omega_1 = \frac{2\pi}{T} \tag{3.68}$$

$$C_0 = \frac{1}{T} \int_{\tau}^{\tau+T} x(t)\,\mathrm{d}t \tag{3.69}$$

$$A_n = \frac{2}{T} \int_{\tau}^{\tau+T} x(t) \cos(n\omega_1 t)\,\mathrm{d}t \tag{3.70}$$

$$B_n = \frac{2}{T} \int_{\tau}^{\tau+T} x(t) \sin(n\omega_1 t)\,\mathrm{d}t \tag{3.71}$$

At a point of discontinuity, say t_0, the series converges to the mean value

$$\frac{x(t_0^+) + x(t_0^-)}{2} \tag{3.72}$$

The series (3.67) with coefficients (3.68)–(3.71) is called the *Fourier series* of the signal $x(t)$, or the *trigonometric Fourier series* of $x(t)$. The coefficients C_0, A_n, and B_n are called the *Fourier coefficients*. The frequency ω_1 is the *fundamental angular frequency*.

The above conditions are known as the *Dirichlet's conditions* and are sufficient, but not necessary, for convergence of Fourier series; these conditions are satisfied by virtually all signals arising in physical and engineering problems.

The kth partial sum of the Fourier series

$$S_k(t) = C_0 + \sum_{n=1}^{k} (A_n \cos(n\omega_1 t) + B_n \sin(n\omega_1 t)) \tag{3.73}$$

is often referred to as the *truncated Fourier series* of $x(t)$.

Computation of the Fourier coefficients can be simplified if $x(t)$ has symmetry. An even signal, $x(-t) = x(t)$, has no sine terms, $B_n = 0$; an odd signal, $x(-t) = -x(t)$, has no cosine terms, $A_n = 0$, and $C_0 = 0$.

The process of resolving a signal into its Fourier series is called *spectral analysis* or *harmonic analysis*.

3.2.2 Complex Form of the Fourier Series

In complex notation, the Fourier series can be written in a more compact representation as follows:

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n \, \mathrm{e}^{jn\omega_1 t} \tag{3.74}$$

where

$$C_n = \frac{1}{T} \int_{\tau}^{\tau+T} x(t) \, \mathrm{e}^{-jn\omega_1 t} \, \mathrm{d}t \tag{3.75}$$

The series (3.74) is the *complex Fourier series* of $x(t)$, also called the *exponential Fourier series*.

Obviously

$$C_{-n} = C_n^* \tag{3.76}$$

and

$$C_n = \frac{A_n - jB_n}{2} \tag{3.77}$$

for $n \neq 0$. (C^* denotes the conjugate of C.)

The complex Fourier coefficients C_n contain all the information about the signal $x(t)$.

3.2.3 Parseval's Identity

Parseval's identity or *Parseval's theorem* states that

$$\frac{1}{T} \int_{\tau}^{\tau+T} |x(t)|^2 \, \mathrm{d}t = C_0^2 + \frac{1}{2} \sum_{n=1}^{+\infty} \left(A_n^2 + B_n^2\right) = \sum_{n=-\infty}^{+\infty} |C_n|^2 \tag{3.78}$$

where A_n and B_n are given by (3.68)–(3.71), and C_n is given by (3.75). As a consequence,

$$\lim_{n\to\infty} A_n = 0 \tag{3.79}$$

$$\lim_{n\to\infty} B_n = 0 \tag{3.80}$$

which is known as the *Riemann's theorem*.

If the signal $x(t)$ represents an electric current or voltage across a resistor, the expression (3.78) is proportional to the resistor average power over a period.

3.2.4 Harmonics of Periodic Signals

The Fourier series (3.67) can be written as

$$x(t) = \sum_{n=0}^{+\infty} x^{(n)}(t) = X^{(0)} + \sum_{n=1}^{+\infty} X_{\mathrm{m}}^{(n)} \cos(n\omega_1 t + \xi^{(n)}) \tag{3.81}$$

where

$$x^{(0)}(t) = X^{(0)} = C_0 \tag{3.82}$$

$$x^{(n)}(t) = X_{\mathrm{m}}^{(n)} \cos(n\omega_1 t + \xi^{(n)}) \tag{3.83}$$

$$X_{\mathrm{m}}^{(n)} = \sqrt{A_n^2 + B_n^2} \tag{3.84}$$

$$\xi^{(n)} = \arg(A_n - jB_n) \tag{3.85}$$

The term $x^{(1)}(t)$ is the *fundamental component* or the first harmonic, $x^{(2)}(t)$ is the second harmonic, and so on. If the signal $x(t)$ represents an electrical quantity, the constant term $X^{(0)}$ is called the *dc component*, and the higher-order harmonics are known as *ac components*; ω_1 is the *fundamental angular frequency*, and $n\omega_1$ is the nth *harmonic angular frequency*.

According to Parseval's identity, the root-mean-square (rms) value of a signal expressed as a Fourier series is the square root of the sum of squares of the rms values of the separate components:

$$X_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_{\tau}^{\tau+T} |x(t)|^2 \,\mathrm{d}t} = \sqrt{\left(X^{(0)}\right)^2 + \frac{1}{2}\sum_{n=1}^{+\infty}\left(X_{\mathrm{m}}^{(n)}\right)^2} \tag{3.86}$$

The quantities $\left(X^{(0)}\right)^2$ and $\frac{1}{2}\left(X_{\mathrm{m}}^{(n)}\right)^2$ show how the power in a periodic signal $x(t)$ is distributed in the frequency domain.

3.2.5 Gibbs Phenomenon

When a sudden change of amplitude occurs in a signal and the attempt is made to represent it by a finite number of terms in a Fourier series, the overshoot at the corners (at the points of abrupt change) is always found. As the number of terms is increased, the overshoot is still found; this is called the *Gibbs phenomenon*.

The best way to illustrate the Gibbs phenomenon is to analyze a unit periodic square pulse train defined as

$$x(t) = \begin{cases} 1, & 0 \le t < \dfrac{T}{2} \\ -1, & \dfrac{T}{2} \le t < T \end{cases}$$
$$x(t + kT) = x(t)$$
$$k = \pm 1, \pm 2, \pm 3, \ldots \tag{3.87}$$

which may be represented by the series

$$x(t) = \sum_{n=1,3,5,\ldots,\infty} \frac{4}{\pi n} \sin\left(2\pi n \frac{t}{T}\right)$$

If we approximate $x(t)$ with a finite sum

$$x_N(t) = \sum_{n=1,3,5,\ldots,N} \frac{4}{\pi n} \sin\left(2\pi n \frac{t}{T}\right)$$

the sum will exhibit an overshoot at points very near the corners $t = k\frac{T}{2}$, and the overshoot will not vanish as N increases—it will have a value of approximately 1.12. [14]

3.2.6 Amplitude and Phase Spectrum of Periodic Signals

A graphical representation of a periodic signal $x(t)$, produced by drawing a series of vertical lines at intervals on a horizontal axis, where the intervals represent an increase of n and the vertical lines are proportional to $X_{\mathrm{m}}^{(n)}$, is called the *amplitude spectrum* of the periodic signal. Often, angular frequencies $n\omega_1$ may replace n as the abscissa.

Similarly, the phase angles $\xi^{(n)}$ plotted against n or $n\omega_1$ are called the *phase spectrum* of the periodic signal. Notice that $\xi^{(0)} \equiv 0$.

The two sequences of numbers

$$\{X^{(0)},\ X_{\mathrm{m}}^{(1)},\ X_{\mathrm{m}}^{(2)},\ \ldots,\ X_{\mathrm{m}}^{(n)},\ \ldots\}$$
$$\{\xi^{(0)},\ \xi^{(1)},\ \xi^{(2)},\ \ldots,\ \xi^{(n)},\ \ldots\}$$

along with the period T, contain all the information embodied in the Fourier series.

The plot of quantities

$$\left(X^{(0)}\right)^2,\ \frac{1}{2}\left(X_{\mathrm{m}}^{(1)}\right)^2,\ \frac{1}{2}\left(X_{\mathrm{m}}^{(2)}\right)^2,\ \ldots,\ \frac{1}{2}\left(X_{\mathrm{m}}^{(n)}\right)^2,\ \ldots \tag{3.88}$$

versus n or $n\omega_1$ is called the *power spectrum*.

The spectra defined above are frequently referred to as the *natural spectra* and are defined for non-negative n.

In considering representations of $x(t)$ in terms of the coefficients C_n the integer n is arbitrary (positive, negative, or zero) and we can define the amplitude spectrum as the plot of $|C_n|$ against n or $n\omega_1$; this spectrum is known as the *mathematical amplitude spectrum*. Similarly, the *mathematical power spectrum* is the plot of $|C_n|^2$ against n or $n\omega_1$ ($n = 0, \pm 1, \pm 2, \pm 3, \ldots$).

The spectra of a periodic signal are *discrete spectra* or *line spectra*.

Sequence of Short Square Pulses. Consider a train of square pulses which occurs at regular intervals T. The amplitude of the pulses is A, and each pulse is of duration τ:

$$x(t) = \begin{cases} A, & 0 < t \leq \tau \\ 0, & \tau < t \leq T \end{cases}$$
$$x(t + kT) = x(t) \tag{3.89}$$
$$k = \pm 1, \pm 2, \pm 3, \ldots$$

The Fourier series for the pulse train $x(t)$ is

$$x(t) = C_0 + \sum_{n=1}^{+\infty} (A_n \cos(n\omega_1 t) + B_n \sin(n\omega_1 t))$$

with

$$C_0 = A\frac{\tau}{T}, \qquad A_n = A\frac{\sin(n\omega_1\tau)}{n\pi}, \qquad B_n = 2A\frac{\sin^2(n\omega_1\tau/2)}{n\pi}, \qquad \omega_1 = \frac{2\pi}{T}$$

or, in the complex form,

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n \, e^{jn\omega_1 t}, \qquad C_n = \frac{A_n - jB_n}{2}$$

A limiting case of great theoretical significance is that of very short pulses ($\tau \to 0$), when pulses are of infinite magnitude and infinitesimal duration, but the product of these quantities is finite, $A\tau \equiv 1$, or $A = 1/\tau$:

$$c(t) = \lim_{\tau \to 0} x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT) = \frac{1}{T} + \frac{2}{T}\sum_{n=1}^{+\infty} \cos(n\omega_1 t) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} e^{jn\omega_1 t} \tag{3.90}$$

The signal $c(t)$ is a train of equidistant Dirac's delta pulses and is frequently called the *comb signal*.

3.2.7 Steady-State Response of a System to a Nonsinusoidal Periodic Stimulus

Consider a single-input single-output lumped LTI system excited by a nonsinusoidal periodic stimulus $x(t)$ resolved into its harmonics (3.81). Again, we focus on the systems with natural frequencies in the open left-half complex plane.

We can find the phasors for each input harmonic component (3.83):

$$X^{(n\omega_1)} = \mathcal{P}_{n\omega_1}(x^{(n)}(t)) = X_{\mathrm{m}}^{(n)} \, e^{j\xi^{(n)}} \tag{3.91}$$

According to the principle of superposition, the steady state resulting from the input harmonics is the sum of the sinusoidal steady states that would exist if each input harmonic were acting alone on the system.

We may use the phasor method to obtain the sinusoidal steady state to a single input harmonic $x^{(n)}(t)$. If the transfer function of the system is known, say $H(j\omega)$, the output harmonic phasors are

$$Y^{(n\omega_1)} = H(jn\omega_1)X^{(n\omega_1)} \tag{3.92}$$

Finally, we determine the steady-state response by taking the inverse phasor transform of the output harmonic phasors

$$y(t) = \sum_{n=0}^{\infty} \mathcal{P}_{n\omega_1}^{-1}(Y^{(n\omega_1)}) \tag{3.93}$$

In other words, the steady state resulting from a nonsinusoidal periodic input is the sum of the sinusoidal steady states that would exist if each input harmonic were acting alone on the system.

3.3 FOURIER TRANSFORM

The Fourier transform occupies so central a place in analysis of signals and systems as to demand at least an introductory treatment. A major reason for this is that the most convenient way of measuring and specifying the performance of a system or signal is based on frequency. If we apply continuous sinusoidal stimuli over a wide range of frequencies and then measure the relation between response and stimuli both in magnitude and phase, the Fourier transform would then make it possible in principle to deduce the response to any other form of stimulus. Therefore, the Fourier transform can be thought of as the ultimate generalization of the phasor transform.

3.3.1 Definition of the Fourier Transform

Consider a nonperiodic signal $x(t)$. The *Fourier transform* of $x(t)$ is

$$X(j\omega) = \mathcal{F}(x(t)) = \int_{-\infty}^{+\infty} x(t)\,\mathrm{e}^{-j\omega t}\,\mathrm{d}t \tag{3.94}$$

and the *inverse Fourier transform* of $X(j\omega)$ is

$$x(t) = \mathcal{F}^{-1}(X(j\omega)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,\mathrm{e}^{j\omega t}\,\mathrm{d}\omega \tag{3.95}$$

The two functions $x(t)$ and $X(j\omega)$ form a *Fourier transform pair* which is designated by $x(t) \leftrightarrow X(j\omega)$. Some authors drop the imaginary unit, j, and denote the Fourier transform of $x(t)$ by $X(\omega)$. The variable ω is called a *continuous frequency variable*.

The Fourier transform exists if

$$\lim_{T\to\infty}\int_{-T}^{T} |x(t)|\,\mathrm{d}t < \infty \tag{3.96}$$

exists. This is a sufficient, but not necessary, condition. The more general version of the existence conditions requires $x(t)$ to have a finite number of maxima and minima, as well as a finite number of discontinuities over the entire range $-\infty < t < \infty$; $x(t)$ may become infinite at some isolated points provided that (3.96) is finite.

The inverse Fourier transform (3.95) gives the value of $x(t)$ in terms of $X(j\omega)$, at any point t where $x(t)$ is continuous. However, at a point of discontinuity, say t_0, (3.95) gives the arithmetic mean $\frac{1}{2}(x(t_0^-) + x(t_0^+))$.

The process through which signals of a real variable t are associated with corresponding complex functions of a new complex variable $j\omega$ is known as a *Fourier transformation*. The complex function $X(j\omega)$ is called the *image* of $x(t)$, and $x(t)$ is known as the *original* of $X(j\omega)$. The process of going back from $X(j\omega)$ to $x(t)$ is referred to as an *inverse Fourier transformation*.

The plots of $|X(j\omega)|$ and $\arg X(j\omega)$ versus ω are called the *amplitude spectrum* and *phase spectrum* of $x(t)$, respectively.

Fourier transform (3.94) can be viewed as the harmonic content per unit interval of frequency $f = \frac{1}{2\pi}\omega$. Thus, it is convenient to call $x(t)$ the signal possessing the spectrum $X(j\omega)$.

Notice that apart from the factor 2π and the sign of the exponents, Eqs. (3.94) and (3.95) are identical in form.

3.3.2 Properties of the Fourier Transform

We summarize the salient properties of the Fourier transform subject always to the proviso that the transform exists.

The Fourier transform is *unique* (except for a finite number of isolated points of discontinuity) for all t:

$$x_1(t) = x_2(t) \Leftrightarrow X_1(j\omega) = X_2(\mathrm{j}\omega) \tag{3.97}$$

where $X_1(j\omega) = \mathcal{F}(x_1(t))$, $X_2(j\omega) = \mathcal{F}(x_2(t))$, and $x_1(t), x_2(t)$ are nonperiodic signals. For physically generated signals and for the signals dealt with in this book, the Fourier transform is unique for all t.

The operators $\mathcal{F}$ and $\mathcal{F}^{-1}$ are *homogeneous*

$$\begin{aligned} \mathcal{F}(Kx(t)) &= K\mathcal{F}(x(t)) \\ \mathcal{F}^{-1}(KX(j\omega)) &= K\mathcal{F}^{-1}(X(j\omega)) \end{aligned} \tag{3.98}$$

for any constant K, with $X(j\omega) = \mathcal{F}(x(t))$.

Also, the two operators are *additive*:

$$\begin{aligned} \mathcal{F}(x_1(t) + x_2(t)) &= \mathcal{F}(x_1(t)) + \mathcal{F}(x_2(t)) \\ \mathcal{F}^{-1}(X_1(j\omega) + X_2(j\omega)) &= \mathcal{F}^{-1}(X_1(j\omega)) + \mathcal{F}^{-1}(X_2(j\omega)) \end{aligned} \tag{3.99}$$

The Fourier transform, $\mathcal{F}$, maps the operation of differentiating, D, into the multiplication by $j\omega$:

$$\begin{aligned}\mathcal{F}(\mathrm{D}x(t)) &= j\omega\mathcal{F}(x(t)) = j\omega X(j\omega)\\ \mathcal{F}^{-1}(j\omega X(j\omega)) &= \mathrm{D}\mathcal{F}^{-1}(X(j\omega)) = \mathrm{D}x(t)\end{aligned} \tag{3.100}$$

where $X(j\omega) = \mathcal{F}(x(t))$, and D denotes differentiation with respect to time, $\mathrm{D}x(t) = \mathrm{d}x(t)/\mathrm{d}t$.

A set of properties (theorems) can be derived from the near-symmetry of the direct and inverse Fourier transformations:

Symmetry

$$\mathcal{F}(X(jt)) = 2\pi x(-\omega) \tag{3.101}$$

$$\mathcal{F}^{-1}(x(\omega)) = \frac{1}{2\pi}X(-jt) \tag{3.102}$$

Time shifting

$$\mathcal{F}(x(t-T)) = X(j\omega)\,\mathrm{e}^{-j\omega T} \tag{3.103}$$

Frequency shifting

$$\mathcal{F}^{-1}(X(j(\omega-\Omega))) = x(t)\mathrm{e}^{j\Omega t} \tag{3.104}$$

Time convolution

$$\mathcal{F}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau = X(j\omega)H(j\omega) \tag{3.105}$$

Frequency convolution

$$\mathcal{F}^{-1}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(jv)Y(j(\omega-v))\,\mathrm{d}v = x(t)y(t) \tag{3.106}$$

where $X(j\omega) = \mathcal{F}(x(t))$, $H(j\omega) = \mathcal{F}(h(t))$, $Y(j\omega) = \mathcal{F}(y(t))$, $T = const$, $\Omega = const$.

Time scaling

$$\mathcal{F}(x(at)) = \frac{1}{|a|}X\left(\frac{j\omega}{a}\right) \tag{3.107}$$

Frequency scaling

$$\mathcal{F}^{-1}(X(ja\omega)) = \frac{1}{|a|}x\left(\frac{t}{a}\right) \tag{3.108}$$

for any real $a \neq 0$.

3.3.3 Convolution

The *convolution* of two signals $x(t)$ and $h(t)$ is denoted by $x(t) * h(t)$ and defined as

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,\mathrm{d}\tau \tag{3.109}$$

The *convolution in the frequency domain* is defined by

$$X(j\omega) * Y(j\omega) = \int_{-\infty}^{\infty} X(jv)Y(j(\omega - v))\,\mathrm{d}v = \int_{-\infty}^{\infty} X(j(\omega - v))Y(jv)\,\mathrm{d}v \quad (3.110)$$

where $X(j\omega)$ and $Y(j\omega)$ are the Fourier transforms of $x(t)$ and $y(t)$, respectively.

Dirac delta impulse, $\delta(t)$, is the *identity element* in the convolution operation

$$x(t) * \delta(t) = x(t) \quad (3.111)$$

Also, $x(t) * h(t) = h(t) * x(t)$, and it can be shown that

$$x(t) * \delta(t - T) = x(t - T) \quad (3.112)$$

for any real time shift T.

The property (3.112) can be useful in representing periodic signals. If the signal $x(t)$ is *time-limited*—that is, it is zero outside a time interval, say $x(t) = 0$ for $|t| > \frac{T}{2}$—then the periodic continuation of $x(t)$, with period T, can be represented as a convolution of $x(t)$ and the comb signal $c(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$:

$$x_{\mathrm{per}}(t) = x(t) * c(t) \quad (3.113)$$

where $x_{\mathrm{per}}(t)$ denotes the periodic signal, and $x(t) = x_{\mathrm{per}}(t)$ for $|t| < \frac{T}{2}$.

Modulation Property. The frequency shifting property can be employed to obtain the Fourier transform of a signal of the form

$$x_{\mathrm{am}}(t) = x(t)\cos(\Omega t) \quad (3.114)$$

in which $x(t)$ is called the *modulating signal*. The term $\cos(\Omega t)$ is said to be *modulated in amplitude*, and is called the *carrier*.

The Fourier transform of $x_{\mathrm{am}}(t)$ can be found by expressing the $\cos(\Omega t)$ in terms of the exponential function

$$\begin{aligned} X_{\mathrm{am}}(j\omega) &= \mathcal{F}(x_{\mathrm{am}}(t)) = \mathcal{F}(x(t)(\mathrm{e}^{j\Omega t} + \mathrm{e}^{-j\Omega t})/2) \\ &= \frac{1}{2}\mathcal{F}(x(t)\,\mathrm{e}^{j\Omega t}) + \frac{1}{2}\mathcal{F}(x(t)\mathrm{e}^{-j\Omega t}) \end{aligned}$$

After employing the frequency shifting property, we find

$$X_{\mathrm{am}}(j\omega) = \frac{1}{2}X(j(\omega - \Omega)) + \frac{1}{2}X(j(\omega + \Omega)) \quad (3.115)$$

where $X(j\omega) = \mathcal{F}(x(t))$ is called the *base-band spectrum*, and the two terms are said to be the *side-bands*.

3.3.4 Parseval's Theorem and Energy Spectral Density

Parseval's formula (theorem) states that

$$\int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, \mathrm{d}\omega \tag{3.116}$$

If $x(t)$ represents an electric voltage or current, then the left-hand integral in (3.116) represents the total energy that would be delivered to a 1-Ω resistor.

The quantity

$$E(\omega) = |X(j\omega)|^2 \tag{3.117}$$

represents the energy per unit bandwidth of frequency (not angular frequency) and is called the *energy spectral density*. A plot of $E(\omega)$ versus ω is known as the *energy spectrum* of $x(t)$. The *total energy* of the signal is

$$W = \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E(\omega) \, \mathrm{d}\omega \tag{3.118}$$

Notice that for a periodic signal the total signal power was obtained, in terms of the Fourier series coefficients, as the sum of the power contents of all the discrete frequency components. For a nonperiodic signal the total signal energy is obtained, in terms of the amplitude spectrum, as the integral of the energy contents of all the continuous frequency components.

Assume that the real signal $x(t)$ has finite energy. The *autocorrelation* function of $x(t)$ is defined as

$$\rho_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau) \, \mathrm{d}t \tag{3.119}$$

It can be shown that the autocorrelation function, $\rho_{xx}(\tau)$, and the energy spectral density, $E(\omega)$, constitute a Fourier transform pair

$$\rho_{xx}(\tau) \leftrightarrow E(\omega) \tag{3.120}$$

that is,

$$\mathcal{F}(\rho_{xx}(\tau)) = E(\omega)$$

or

$$\rho_{xx}(\tau) = \mathcal{F}^{-1}(E(\omega))$$

which is known as the *Wiener–Kintchine theorem*.

3.3.5 Properties of the Fourier Transform of Real Signals

Assume that the signal $x(t)$ is a real function of time—that is, of the real variable t—and that its Fourier transform $X(j\omega)$ may be expressed as either

$$X(j\omega) = M_x(\omega)\, \mathrm{e}^{j\Phi_x(\omega)}$$

or

$$X(j\omega) = X_{\mathrm{re}}(\omega) + jX_{\mathrm{im}}(\omega)$$

where $M_x(\omega) = |X(j\omega)|$, $\Phi_x(\omega) = \arg(X(j\omega))$, $X_{\mathrm{re}}(\omega) = \mathrm{Re}(X(j\omega))$, and $X_{\mathrm{im}}(\omega) = \mathrm{Im}(X(j\omega))$.

From the definition of the Fourier transform (3.94) the real and imaginary parts of $X(j\omega)$ become, respectively,

$$X_{\mathrm{re}}(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)\,\mathrm{d}t \tag{3.121}$$

$$X_{\mathrm{im}}(\omega) = -\int_{-\infty}^{\infty} x(t)\sin(\omega t)\,\mathrm{d}t \tag{3.122}$$

It can be shown (after employing the definition and/or the properties of the Fourier transform) that the following holds:

- The real part of $X(j\omega)$ is an **even** function of ω, $X_{\mathrm{re}}(-\omega) = X_{\mathrm{re}}(\omega)$.
- The imaginary part of $X(j\omega)$ is an **odd** function of ω, $X_{\mathrm{im}}(-\omega) = -X_{\mathrm{im}}(\omega)$.
- Changing the sign of ω in $X(j\omega)$ is equivalent to taking the complex conjugate of $X(j\omega)$, $X(-j\omega) = X^*(j\omega)$ (the asterisk denotes the complex conjugate).
- The amplitude spectrum is an **even** function of ω, $M_x(-\omega) = M_x(\omega)$.
- The phase spectrum is an **odd** function of ω, $\Phi_x(-\omega) = -\Phi_x(\omega)$.
- The energy spectral density is an **even** function of ω, $E(-\omega) = E(\omega)$.
- If $x(t)$ is an **even** function of t, $X(j\omega)$ is **real**, $X(j\omega) = X_{\mathrm{re}}(\omega)$.
- If $x(t)$ is an **odd** function of t, $X(j\omega)$ is **imaginary**, $X(j\omega) = jX_{\mathrm{im}}(\omega)$.
- Changing the sign of t in $x(t)$ is equivalent to changing the sign of ω in $X(j\omega)$, $\mathcal{F}(x(-t)) = X(-j\omega)$, implying that $x(-t)$ and $X(-j\omega)$ constitute a Fourier transformation pair, $x(-t) \leftrightarrow X(-j\omega)$, or $x(-t) \leftrightarrow X^*(j\omega)$.

The real signal $x(t)$ can be decomposed into the sum of its even part

$$\mathrm{Ev}(x(t)) = \frac{x(t) + x(-t)}{2} \tag{3.123}$$

and its odd part

$$\mathrm{Od}(x(t)) = \frac{x(t) - x(-t)}{2} \tag{3.124}$$

that is,

$$x(t) = \mathrm{Ev}(x(t)) + \mathrm{Od}(x(t))$$

The even and the odd parts of $x(t)$ form Fourier pairs with the real and the imaginary parts of the Fourier transform of $x(t)$:

$$\mathrm{Ev}(x(t)) \leftrightarrow X_{\mathrm{re}}(\omega)$$
$$\mathrm{Od}(x(t)) \leftrightarrow jX_{\mathrm{im}}(\omega)$$

A number of useful Fourier transform pairs for some real signals follows

$$
\begin{array}{ccc}
x(t) & \leftrightarrow & X(j\omega) \\
\delta(t) & \leftrightarrow & 1 \\
1 & \leftrightarrow & 2\pi\delta(\omega) \\
\cos(\Omega t) & \leftrightarrow & \pi\delta(\omega+\Omega)+\pi\delta(\omega-\Omega) \\
\sin(\Omega t) & \leftrightarrow & j\pi\delta(\omega+\Omega)-\mathrm{j}\pi\delta(\omega-\Omega) \\
\mathrm{e}^{j\Omega t} & \leftrightarrow & 2\pi\delta(\omega-\Omega) \\
p_T(t)=\begin{cases}1, & |t|\le T\\ 0, & |t|>T\end{cases} & \leftrightarrow & \frac{2}{\omega}\sin(\omega T) \\
\frac{1}{\pi t}\sin(\Omega t) & \leftrightarrow & p_\Omega(\omega)=\begin{cases}1, & |\omega|\le\Omega\\ 0, & |\omega|>\Omega\end{cases} \\
u(t) & \leftrightarrow & \frac{1}{j\omega}+\pi\delta(\omega) \\
\mathrm{e}^{-at}u(t) & \leftrightarrow & \frac{1}{a+j\omega}
\end{array}
\tag{3.125}
$$

The pairs are generated by employing the definition of the transform or its properties. A thorough treatment of the Fourier transform of special signals, like Dirac delta $\delta(t)$, unity function $x(t)=1$, and unit step $u(t)$ (i.e., generalized functions or distributions), is not given here; it can be found in references 5 and 14.

3.3.6 Causal Signals and the Hilbert Transform

Time-varying signals that can be generated by physically realizable signal sources are causal and do not exist prior to $t=t_0$. By choosing the convenient time origin we can always adjust $t_0=0$. Therefore, we define a *causal signal* as a signal which exists only for positive values of time and is equal to zero otherwise:

$$x(t)=0, \qquad t<0 \tag{3.126}$$

An important property of a real causal signal is that it can be uniquely determined from either the real part or the imaginary part of its Fourier transform:

$$x(t)=\frac{2}{\pi}\int_0^\infty X_{\mathrm{re}}(\omega)\cos(\omega t)\,\mathrm{d}t \tag{3.127}$$

$$x(t)=-\frac{2}{\pi}\int_0^\infty X_{\mathrm{im}}(\omega)\sin(\omega t)\,\mathrm{d}t \tag{3.128}$$

Also, we can write

$$x(t)\leftrightarrow 2X_{\mathrm{re}}(\omega)$$

$$x(t)\leftrightarrow j2X_{\mathrm{im}}(\omega)$$

or equivalently

$$\mathcal{F}(x(t)) = 2X_{\text{re}}(\omega)$$
$$\mathcal{F}(x(t)) = j2X_{\text{im}}(\omega)$$

The real and imaginary part of a real causal signal are related by integrals of the form

$$X_{\text{re}}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{X_{\text{im}}(v)}{\omega - v}\, \mathrm{d}v \tag{3.129}$$

$$X_{\text{im}}(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{X_{\text{re}}(v)}{\omega - v}\, \mathrm{d}v \tag{3.130}$$

The expressions (3.129) and (3.130) are known as the *Hilbert transform pair*.

3.3.7 Application of the Fourier Transform

The Fourier transformation can be used in the computation of the solution of ordinary linear differential equations with real constant coefficients when the forcing function is nonperiodic and when all initial conditions are zero; this computation is referred to as the *Fourier transform method*.

Consider a lumped LTI system with a single input $x(t)$ and a single output $y(t)$, described by the following differential equation:

$$\begin{aligned} a_n \mathrm{D}^n y + a_{n-1}\mathrm{D}^{n-1} y + \cdots + a_1 \mathrm{D} y + a_0 y \\ = b_m \mathrm{D}^m x + b_{m-1}\mathrm{D}^{m-1} x + \cdots + b_1 \mathrm{D} x + b_0 x \end{aligned} \tag{3.131}$$

that is,

$$\sum_{i=0}^{n} a_i \mathrm{D}^i y = \sum_{k=0}^{m} b_k \mathrm{D}^k x \tag{3.132}$$

or, by introducing the polynomial differential operators,

$$A(\mathrm{D})\, y = B(\mathrm{D})\, x \tag{3.133}$$

where $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ are real numbers.

We apply the Fourier transform (3.94) to both sides of Eq. (3.132) to obtain

$$\mathcal{F}\left(\sum_{i=0}^{n} a_i \mathrm{D}^i y\right) = \mathcal{F}\left(\sum_{k=0}^{m} b_k \mathrm{D}^k x\right)$$

From the additivity (3.99) and homogeneity (3.98) property this becomes

$$\sum_{i=0}^{n} a_i \mathcal{F}\left(\mathrm{D}^i y\right) = \sum_{k=0}^{m} b_k \mathcal{F}\left(\mathrm{D}^k x\right)$$

and from the differentiating property (3.100), applied repeatedly, we get

$$\sum_{i=0}^{n} a_i\,(j\omega)^i Y(j\omega) \;=\; \sum_{k=0}^{m} b_k\,(j\omega)^k X(j\omega) \tag{3.134}$$

with $Y(j\omega) = \mathcal{F}(y(t))$, or, equivalently,

$$\left(\sum_{i=0}^{n} a_i\,(j\omega)^i\right) Y(j\omega) \;=\; \left(\sum_{k=0}^{m} b_k\,(j\omega)^k\right) X(j\omega)$$

By using the compact notation similar to Eq. (3.133) we may write

$$A(j\omega)Y(j\omega) \;=\; B(j\omega)X(j\omega) \tag{3.135}$$

Thus, if

$$A(j\omega) \;\neq\; 0 \tag{3.136}$$

we compute the Fourier transform of the system output

$$Y(j\omega) \;=\; \frac{B(j\omega)}{A(j\omega)} X(j\omega) \tag{3.137}$$

where

$$A(j\omega) \;=\; \sum_{i=0}^{n} a_i\,(j\omega)^i \tag{3.138}$$

$$B(j\omega) \;=\; \sum_{k=0}^{m} b_k\,(j\omega)^k \tag{3.139}$$

Equation (3.134) can be obtained directly from Eq. (3.132) by replacing the ith derivatives of $y(t)$ with $(j\omega)^i Y(j\omega)$, for $i = 0$ to n, and by replacing the kth derivatives of $x(t)$ with $(j\omega)^k X(j\omega)$, for $k = 0$ to m. Also, we can write Eq. (3.135) directly from Eq. (3.133) by replacing the differential operator D with $j\omega$, $x(t)$ with $X(j\omega)$, and $y(t)$ with $Y(j\omega)$. Note that polynomial differential operators $A(\mathrm{D})$ and $B(\mathrm{D})$ are mapped into algebraic polynomials in complex variable $j\omega$ [Eqs. (3.138) and (3.139)].

The solution is found by the inverse Fourier transform (3.95)

$$y(t) \;=\; \mathcal{F}^{-1}(Y(j\omega)) \;=\; \mathcal{F}^{-1}(H(j\omega)X(j\omega)) \tag{3.140}$$

where

$$H(j\omega) \;=\; \frac{B(j\omega)}{A(j\omega)} \tag{3.141}$$

is the transfer function of the system. The response $y(t)$ is the zero-state response, $y(t) = y_x(t)$, and is excited solely by the input signal $x(t)$.

If the excitation is the Dirac delta, $x(t) = \delta(t)$, whose Fourier transform is unity, $X(j\omega) = \mathcal{F}(\delta(t)) = 1$, then the response, $y_\delta(t)$, can be obtained from (3.140) as

$$y(t) \;=\; y_\delta(t) \;=\; \mathcal{F}^{-1}(H(j\omega)X(j\omega)) \;=\; \mathcal{F}^{-1}(H(j\omega)) \tag{3.142}$$

It follows that the impulse response and the transfer function constitute Fourier transform pair. It is customarily to designate the impulse response by $h(t)$, so

$$h(t) \leftrightarrow H(j\omega) \tag{3.143}$$

Knowledge of the transfer function allows us to find the zero-state response due to any type of excitation. For the given transfer function $H(j\omega)$ and the input $x(t)$, we can find the zero-state response from

$$y_x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau, \qquad h(t) = \mathcal{F}^{-1}(H(j\omega)) \tag{3.144}$$

Generally, it is more efficient to compute the Fourier transform of the input signal, multiply it by the transfer function, and take the inverse Fourier transform to find the zero-state response, that is,

$$y_x(t) = \mathcal{F}^{-1}(H(j\omega)\mathcal{F}(x(t))) \tag{3.145}$$

The impulse response $h(t)$ is a real signal if the system under consideration is real. According to the properties of the Fourier transform of real signals, it follows that the magnitude response, $M(\omega) = |H(j\omega)|$, is an even function of ω, while the phase response, $\Phi(\omega) = \arg(H(j\omega))$, is an odd function of ω:

$$M(-\omega) = M(\omega)$$
$$\Phi(-\omega) = -\Phi(\omega)$$

If we can compute (or measure) the transfer function for $\omega \geq 0$, we can find it for $\omega < 0$, too.

A system is said to be *causal* if its impulse response is a causal signal:

$$h(t) = 0, \qquad t < 0 \tag{3.146}$$

If we excite a relaxed causal system by a causal input signal

$$x(t) = 0, \qquad t < 0$$

the zero-state response will be causal:

$$y_x(t) = 0, \qquad t < 0$$

In addition, the real part and the imaginary part of the transfer function of a causal system are not independent, because they form a Hilbert transform pair:

$$H_{\mathrm{re}}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{H_{\mathrm{im}}(v)}{\omega - v}\,\mathrm{d}v \tag{3.147}$$

$$H_{\mathrm{im}}(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{H_{\mathrm{re}}(v)}{\omega - v}\,\mathrm{d}v \tag{3.148}$$

where $H_{\mathrm{re}}(\omega) = \mathrm{Re}(H(j\omega))$ and $H_{\mathrm{im}}(\omega) = \mathrm{Im}(H(j\omega))$.

3.3.8 Fourier Transform of Sampled Signals and the Sampling Theorem

The infinite train of Dirac delta impulses, which we call the comb signal, can be expressed as

$$c(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT) = \frac{1}{2\pi}\Omega \sum_{n=-\infty}^{+\infty} \mathrm{e}^{jn\Omega t}, \qquad \Omega = \frac{2\pi}{T} \tag{3.149}$$

and is useful in describing the important operation of sampling a continuous signal.

The Fourier transform of the comb signal can be found as

$$C(j\omega) = \mathcal{F}(c(t)) = \frac{1}{2\pi}\Omega \sum_{n=-\infty}^{+\infty} \mathcal{F}(\mathrm{e}^{jn\Omega t}) = \frac{1}{2\pi}\Omega \sum_{n=-\infty}^{+\infty} 2\pi\delta(\omega - n\Omega)$$

that is,

$$C(j\omega) = \Omega \sum_{n=-\infty}^{+\infty} \delta(\omega - n\Omega)$$

or

$$\begin{gathered}\sum_{k=-\infty}^{+\infty} \delta(t - kT) \leftrightarrow \Omega \sum_{n=-\infty}^{+\infty} \delta(\omega - n\Omega) \\ \Omega T = 2\pi\end{gathered} \tag{3.150}$$

In other words, the Fourier transform of an infinite train of equidistant Dirac impulses is another infinite train of equidistant Dirac impulses.

A *sampled signal*, denoted as $x_{\mathrm{samp}}(t)$, can be generated by multiplying a continuous-time signal $x(t)$ by the comb signal $c(t)$:

$$x_{\mathrm{samp}}(t) = x(t)c(t)$$

The comb signal acts as an ideal impulse sampler and transforms $x(t)$ into a sequence of Dirac impulses, and each impulse is of strength $x(kT)$.

The sampled signal can be expressed in the form

$$x_{\mathrm{samp}}(t) = \sum_{k=-\infty}^{+\infty} x(kT)\delta(t - kT) = \frac{1}{2\pi}\Omega \sum_{n=-\infty}^{+\infty} x(t)\,\mathrm{e}^{jn\Omega t}, \qquad \Omega = \frac{2\pi}{T}$$

and its Fourier transform is

$$X_{\mathrm{samp}}(j\omega) = \mathcal{F}(x_{\mathrm{samp}}(t)) = \sum_{k=-\infty}^{+\infty} x(kT)\mathrm{e}^{-jk\omega T} = \frac{1}{2\pi}\Omega \sum_{n=-\infty}^{+\infty} X(j(\omega - n\Omega)) \tag{3.151}$$

The spectrum of the sampled signal, $X_{\mathrm{samp}}(j\omega)$, can be viewed as a periodic continuation of the base-band spectrum $X(j\omega)$.

The equation (3.151) can be rewritten, for $\omega = 0$, as *Poisson's summation formula* (theorem):

$$\sum_{n=-\infty}^{\infty} x(nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(jn\Omega) \tag{3.152}$$

with $\Omega = \frac{2\pi}{T}$. If the signal is causal, $x(t) = 0$ for $t < 0$, then

$$x(0^+) + \sum_{n=1}^{\infty} x(nT) = \frac{x(0^+)}{2} + \frac{1}{T} \sum_{n=-\infty}^{\infty} X(jn\Omega) \tag{3.153}$$

where $x(0^+) = \lim_{\substack{t \to 0 \\ t > 0}} x(t)$.

For many practical signals the spectrum may be zero outside a certain frequency band

$$X(j\omega) = 0, \qquad |\omega| > \Omega_x \tag{3.154}$$

Such signals are called *band-limited signals*.

The *sampling theorem* states that a band-limited signal $x(t)$ for which

$$X(j\omega) = 0, \qquad |\omega| \geq \frac{1}{2}\Omega \tag{3.155}$$

can be uniquely determined from its samples $x(kT)$, where $\Omega T = 2\pi$. The frequency at which we take samples is called the *sampling frequency*, $f_{\text{samp}} = \frac{1}{T} = \frac{\Omega}{2\pi}$, and must be greater than or equal to twice the maximum frequency in the spectrum of $x(t)$:

$$f_{\text{samp}} \geq 2F_x = \frac{\Omega_x}{\pi} \tag{3.156}$$

The sampled signal can be converted back into the original continuous-time signal if we pass the sampled signal through a system characterized by the transfer function

$$\begin{aligned} H(j\omega) &= T, \qquad |\omega| < \frac{\Omega}{2} \\ H(j\omega) &= 0, \qquad |\omega| \geq \frac{\Omega}{2} \end{aligned} \tag{3.157}$$

Such a system is called an *ideal lowpass filter*. If the sampling frequency is lower than required by (3.156), the output of the ideal lowpass filter will at best yield a distorted version of $x(t)$; this effect is called *aliasing* or *frequency folding*.

3.3.9 Fourier Transform of Periodic Signals and Power Spectral Density

Consider a real periodic signal $x_{\text{per}}(t)$ with period T. According to (3.113) the periodic signal can be expressed as a convolution

$$x_{\text{per}}(t) = x(t) * c(t) \tag{3.158}$$

of a nonperiodic time-limited signal $x(t)$, which coincides with $x_{\text{per}}(t)$ over one period $|t| < \frac{T}{2}$,

$$x(t) = \begin{cases} x_{\text{per}}(t), & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & |t| \geq \frac{T}{2} \end{cases} \tag{3.159}$$

and the comb signal (3.149)

$$c(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT) \tag{3.160}$$

The Fourier transform of the periodic signal $x_{\text{per}}(t)$ can be found by using the time-convolution property (3.105) and the Fourier transform of the comb function (3.150):

$$X_{\text{per}}(j\omega) = \mathcal{F}(x_{\text{per}}(t)) = \mathcal{F}(x(t) * c(t)) = X(j\omega)C(j\omega)$$

that is,

$$X_{\text{per}}(j\omega) = X(j\omega)\Omega \sum_{n=-\infty}^{+\infty} \delta(\omega - n\Omega) = \Omega \sum_{n=-\infty}^{+\infty} X(jn\Omega)\delta(\omega - n\Omega)$$

with $\Omega = \frac{2\pi}{T}$.

The signal $x(t)$ is time-limited, so its Fourier transform becomes

$$X(j\omega) = \mathcal{F}(x(t)) = \int_{-T/2}^{T/2} x(t)\mathrm{e}^{-j\omega t}\, \mathrm{d}t \tag{3.161}$$

We may write

$$X_{\text{per}}(j\omega) = \sum_{n=-\infty}^{+\infty} \frac{X(jn\Omega)}{T} 2\pi\delta(\omega - n\Omega) \tag{3.162}$$

and

$$x_{\text{per}}(t) = \mathcal{F}^{-1}(X_{\text{per}}(j\omega)) = \sum_{n=-\infty}^{+\infty} \frac{X(jn\Omega)}{T} \mathcal{F}^{-1}(2\pi\delta(\omega - n\Omega)) \tag{3.163}$$

or

$$x_{\text{per}}(t) = \sum_{n=-\infty}^{+\infty} \frac{X(jn\Omega)}{T}\, \mathrm{e}^{jn\Omega t} \tag{3.164}$$

Obviously, the expression $\frac{X(jn\Omega)}{T}$ is the coefficient C_n, given by (3.75), in the complex form of the Fourier series (3.74) of the periodic signal $x_{\text{per}}(t)$:

$$C_n = \frac{X(jn\Omega)}{T} = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\mathrm{e}^{-jn\Omega t}\, \mathrm{d}t \tag{3.165}$$

with the fundamental angular frequency $\omega_1 = \Omega$.

We conclude from (3.162) that the Fourier transform of a periodic function is an infinite train of equidistant Dirac delta impulses at the harmonic frequencies; the strength of impulses is proportional to the complex coefficients of the Fourier series. Therefore, periodic signals have discrete Fourier transform spectra.

We define the *power spectral density* of a periodic signal as

$$P_{\mathrm{per}}(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} |C_n|^2 \delta(\omega - n\Omega) \tag{3.166}$$

The plot of $P_{\mathrm{per}}(\omega)$ against ω is called the *power spectrum* of a periodic signal.

The *autocorrelation function of periodic signals* is defined by

$$\rho_{\mathrm{per}}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x_{\mathrm{per}}(t) x_{\mathrm{per}}(t + \tau)\, \mathrm{d}t \tag{3.167}$$

and can be expressed as

$$\rho_{\mathrm{per}}(\tau) = \sum_{n=-\infty}^{+\infty} |C_n|^2\, \mathrm{e}^{jn\Omega\tau} \tag{3.168}$$

The quantity $\rho_{\mathrm{per}}(0)$ is equal to the *average power* of the periodic signal

$$P_{\mathrm{av}} = \frac{1}{T} \int_{-T/2}^{T/2} x_{\mathrm{per}}^2(t)\, \mathrm{d}t \tag{3.169}$$

computed over one period.

The power spectral density and the autocorrelation function of a periodic signal form a Fourier transform pair

$$\rho_{\mathrm{per}}(\tau) \leftrightarrow P_{\mathrm{per}}(\omega) \tag{3.170}$$

which is the Wiener–Kintchine theorem for periodic signals.

3.3.10 Fourier Transform and the Phasor Method

Consider a relaxed, single-input single-output, lumped LTI causal system characterized by its transfer function

$$H(j\omega) = M(\omega)\mathrm{e}^{j\Phi(\omega)}$$

and assume that the excitation to the system is sinusoidal:

$$x(t) = X_{\mathrm{m}} \cos(\omega_x t + \xi)$$

for all t.

The zero-state response of the system can be obtained by (3.145)

$$y_x(t) = \mathcal{F}^{-1}(H(j\omega)\mathcal{F}(x(t)))$$

from which it follows

$$\begin{aligned}
y_x(t) &= \mathcal{F}^{-1}(H(j\omega)\mathcal{F}(X_{\mathrm{m}}\cos(\omega_x t + \xi))) \\
&= X_{\mathrm{m}}\mathcal{F}^{-1}(H(j\omega)\mathcal{F}(\mathrm{e}^{j\omega_x t + j\xi} + \mathrm{e}^{-j\omega_x t - j\xi})/2) \\
&= \frac{1}{2}X_{\mathrm{m}}\mathcal{F}^{-1}(H(j\omega)\mathrm{e}^{j\xi}\,\mathcal{F}(\mathrm{e}^{j\omega_x t}) + H(j\omega)\mathrm{e}^{-j\xi}\,\mathcal{F}(\mathrm{e}^{-j\omega_x t}))
\end{aligned}$$

By using the Fourier pair $\mathcal{F}(\mathrm{e}^{j\omega_x t}) = 2\pi\delta(\omega - \omega_x)$ we obtain

$$y_x(t) = \frac{1}{2}X_{\mathrm{m}}\mathcal{F}^{-1}(H(j\omega)\mathrm{e}^{j\xi}\,2\pi\delta(\omega - \omega_x) + H(j\omega)\mathrm{e}^{-j\xi}\,2\pi\delta(\omega + \omega_x))$$

Since $f(t)\delta(t - T) = f(T)\delta(t - T)$, we find

$$\begin{aligned}
y_x(t) &= \frac{1}{2}X_{\mathrm{m}}\mathcal{F}^{-1}(H(j\omega_x)\mathrm{e}^{j\xi}\,2\pi\delta(\omega - \omega_x) + H(-j\omega_x)\mathrm{e}^{-j\xi}\,2\pi\delta(\omega + \omega_x)) \\
&= \frac{1}{2}X_{\mathrm{m}}H(j\omega_x)\mathrm{e}^{j\xi}\,\mathrm{e}^{j\omega_x t} + \frac{1}{2}X_{\mathrm{m}}H(-j\omega_x)\mathrm{e}^{-j\xi}\,\mathrm{e}^{-j\omega_x t}
\end{aligned}$$

The magnitude response is an even function in ω, $M(-\omega) = M(\omega)$, and the phase response is an odd function in ω, $\Phi(-\omega) = -\Phi(\omega)$, so

$$\begin{aligned}
y_x(t) &= \frac{1}{2}X_{\mathrm{m}}M(\omega_x)\mathrm{e}^{j(\omega_x t + \xi + \Phi(\omega_x))} + \frac{1}{2}X_{\mathrm{m}}M(\omega_x)\mathrm{e}^{-j(\omega_x t + \xi + \Phi(\omega_x))} \\
&= \mathrm{Re}(X_{\mathrm{m}}M(\omega_x)\mathrm{e}^{j(\omega_x t + \xi + \Phi(\omega_x))})
\end{aligned}$$

Finally,

$$y_x(t) = X_{\mathrm{m}}M(\omega_x)\cos(\omega_x t + \xi + \Phi(\omega_x)) \tag{3.171}$$

under the condition that $M(\omega_x)$ and $\Phi(\omega_x)$ exist at ω_x.

The response $y_x(t)$ is the steady-state sinusoidal response and is exactly the same as the result obtained by the phasor method:

$$y_x(t) = \mathcal{P}_{\omega_x}^{-1}(H(j\omega_x)X^{(\omega_x)}), \quad X^{(\omega_x)} = X_{\mathrm{m}}\,\mathrm{e}^{j\xi}$$

Therefore, the Fourier transform can be thought of as the ultimate generalization of the phasor transform.

3.4 LAPLACE TRANSFORM

The Laplace transform is perhaps the mathematical signature of the electrical engineer, having a long history of application to problems of electrical engineering.

The term *transform* refers to a mathematical operation that takes a given function and returns a new function. The transformation is often done by means of an integral formula. Commonly used transforms are named after Laplace and Fourier. Transforms are frequently used to change a complicated problem into a simpler one. The simpler problem is then solved, usually using the inverse transform. A standard example is the use of the Laplace transform to solve a differential equation.

The Laplace transform is particularly useful as an analytical tool in the analysis, characterization, and study of linear time-invariant (LTI) systems. It plays a particularly important role in analyzing causal systems specified by linear constant-coefficient differential equations with nonzero initial conditions (i.e., which are not initially at rest).

3.4.1 Definition of the Laplace Transform

The *unilateral Laplace transform* $X(s)$ of a signal $x(t)$ is defined as

$$X(s) = \mathcal{L}(x(t)) = \int_0^\infty x(t)\mathrm{e}^{-st}\,\mathrm{d}t. \tag{3.172}$$

The range of values s for which the integral in Eq. (3.172) converges is referred to as the *region of convergence* (ROC) of the Laplace transform. The complex variable s is in general of the form $s = \sigma + j\omega$, with σ and ω the real and imaginary parts, respectively.

The *inverse Laplace transform* is

$$x(t) = \mathcal{L}^{-1}(X(s)) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\mathrm{e}^{st}\,\mathrm{d}s. \tag{3.173}$$

The contour of integration is a straight line in the complex plane, parallel to the $j\omega$-axis and determined by any value of σ so that $X(\sigma + j\omega)$ converges. For the class of rational transforms, the inverse Laplace transform can be determined without direct evaluation of the integral in Eq. (3.173) by utilizing the partial fraction expansion. [3] Basically, the procedure consists of expanding the rational algebraic expression into a linear combination of lower-order terms of the same type.

The two functions $x(t)$ and $X(s)$ form a *Laplace transform pair* which is designated by $x(t) \leftrightarrow X(s)$. The variable s is called a *complex frequency variable*.

The process through which signals of a real variable t are associated with corresponding complex functions of a new complex variable s is known as a *Laplace transformation*. The complex function $X(s)$ is called the *image* of $x(t)$, and $x(t)$ is known as the *original* of $X(s)$. The process of going back from $X(s)$ to $x(t)$ is referred to as an *inverse Laplace transformation*.

3.4.2 Properties of the Laplace Transform

We summarize the salient properties of the Laplace transform that is always subject to the proviso that the transform exists.

Uniqueness. The Laplace transform is *unique* (except for a finite number of isolated points of discontinuity) for all t:

$$x_1(t) = x_2(t) \Leftrightarrow X_1(s) = X_2(s) \tag{3.174}$$

where $X_1(s) = \mathcal{L}(x_1(t))$, $X_2(s) = \mathcal{L}(x_2(t))$. For physically generated signals and for the signals dealt with in this book, the Laplace transform is unique for all t.

Homogeneity. The operators $\mathcal{L}$ and $\mathcal{L}^{-1}$ are *homogeneous*:

$$\begin{aligned} \mathcal{L}(Kx(t)) &= K\mathcal{L}(x(t)) \\ \mathcal{L}^{-1}(KX(s)) &= K\mathcal{L}^{-1}(X(s)) \end{aligned} \tag{3.175}$$

for any constant K, with $X(s) = \mathcal{L}(x(t))$.

Additivity. Also, the two operators are *additive*:

$$\begin{aligned} \mathcal{L}(x_1(t) + x_2(t)) &= \mathcal{L}(x_1(t)) + \mathcal{L}(x_2(t)) \\ \mathcal{L}^{-1}(X_1(s) + X_2(s)) &= \mathcal{L}^{-1}(X_1(s)) + \mathcal{L}^{-1}(X_2(s)) \end{aligned} \tag{3.176}$$

Differentiation. The Laplace transform, $\mathcal{L}$, maps the operation of differentiating, D, into the multiplication by s:

$$\mathcal{L}(\mathrm{D}x(t)) = s\mathcal{L}(x(t)) - x(0^-) = sX(s) - x(0^-) \tag{3.177}$$

where $X(s) = \mathcal{L}(x(t))$, and D denotes differentiation with respect to time, $\mathrm{D}x(t) = \mathrm{d}x(t)/\mathrm{d}t$.

Convolution. The *convolution* of two causal signals $x(t)$ and $h(t)$ is denoted by $x(t) * h(t)$ and defined as

$$x(t) * h(t) = \int_0^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau = \int_0^{\infty} x(t-\tau)h(\tau)\,\mathrm{d}\tau \tag{3.178}$$

Dirac delta impulse, $\delta(t)$, is the *identity element* in the convolution operation

$$x(t) * \delta(t) = x(t) \tag{3.179}$$

Also, $x(t) * h(t) = h(t) * x(t)$, and it can be shown that

$$x(t) * \delta(t-T) = x(t-T) \tag{3.180}$$

for any real time shift T.

The convolution property of the Laplace transform is

$$\mathcal{L}\int_0^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau = X(s)H(s) \tag{3.181}$$

The Laplace transform is particularly useful as an analytical tool in the analysis, characterization, and study of causal linear time-invariant (LTI) systems excited by causal signals. Its role for this class of systems stems directly from the convolution property of the transform, from which it follows that the Laplace transform of the input and output of an LTI system are related through multiplication by the Laplace transform of the system impulse response (assuming zero initial conditions). Thus, $Y(s) = H(s)X(s)$, where $X(s)$, $Y(s)$, and $H(s)$ are the Laplace transforms of the system input, output, and impulse response, respectively.

3.4.3 The Inverse Laplace Transform of Rational Functions

We shall be concerned with *proper* rational functions $X(s)$ with real coefficients. Specifically, suppose that the denominator of $X(s)$ has distinct roots $p_1, p_2, \dots, p_r$ with *multiplicities* $m_1, m_2, \dots, m_r$. In this case the inverse Laplace transform of $X(s)$ is of the form

$$x(t) = \mathcal{L}^{-1}(X(s)) = \sum_{i=1}^{r} \sum_{k=1}^{m_i} A_{ik} \frac{t^{k-1}}{(k-1)!} \mathrm{e}^{p_i t} \tag{3.182}$$

where the A_{ik} are computed from the equation

$$A_{ik} = \frac{1}{(m_i - k)!} \lim_{s \to p_i} \left(\frac{\mathrm{d}^{m_i - k}}{\mathrm{d}s^{m_i - k}} \left((s - p_i)^{m_i} X(s) \right) \right) \tag{3.183}$$

We use the factorial notation $n!$ for the product $n(n-1)(n-2)\dots 2 \cdot 1$; the quantity $0!$ is defined to be equal to 1.

Standard Laplace Transform Pairs. A number of useful Laplace transform pairs for some real causal signals $x(t)$ follows:

$$\begin{array}{lcl}
x(t) & \leftrightarrow & X(s) \\
\delta(t) & \leftrightarrow & 1 \\
u(t) & \leftrightarrow & \dfrac{1}{s} \\
\mathrm{e}^{-at}\, u(t) & \leftrightarrow & \dfrac{1}{s + a} \\
t^n\, u(t) & \leftrightarrow & \dfrac{n!}{s^{n+1}} \\
\sin(\omega t)\, u(t) & \leftrightarrow & \dfrac{\omega}{s^2 + \omega^2} \\
\cos(\omega t)\, u(t) & \leftrightarrow & \dfrac{s}{s^2 + \omega^2} \\
\mathrm{e}^{-\alpha t} \sin(\omega t)\, u(t) & \leftrightarrow & \dfrac{\omega}{(s + \alpha)^2 + \omega^2} \\
\mathrm{e}^{-\alpha t} \cos(\omega t)\, u(t) & \leftrightarrow & \dfrac{(s + \alpha)}{(s + \alpha)^2 + \omega^2}
\end{array} \tag{3.184}$$

The pairs are generated by employing the definition of the transform or its properties.

3.4.4 Transfer Function of Continuous-Time Systems

Consider a relaxed, single-input, single-output, continuous-time LTI system described by means of a linear constant-coefficients differential equation:

$$\sum_{m=0}^{M} a_m \mathrm{D}^m y(t) = \sum_{l=0}^{L} b_l \mathrm{D}^l x(t) \tag{3.185}$$

Assume that the system is excited by an input causal signal $x(t)$, and observe the output signal $y(t)$. By applying the Laplace transform to both sides of the differential equation, and after employing the linearity property and the differentiating property, we obtain an equation relating the Laplace transforms of the two signals:

$$\left(\sum_{m=0}^{M} a_m s^m\right) Y(s) = \left(\sum_{l=0}^{L} b_l s^l\right) X(s)$$

The system transforms the input signal by multiplying the Laplace transform of the input signal with the factor

$$H(s) = \frac{\sum_{l=0}^{L} b_l s^l}{\sum_{m=0}^{M} a_m s^m} \tag{3.186}$$

The function $H(s)$ is a rational function in s:

$$H(s) = \frac{B(s)}{A(s)}, \qquad B(s) = \sum_{l=0}^{L} b_l s^l, \qquad A(s) = \sum_{m=0}^{M} a_m s^m$$

where $A(s)$ and $B(s)$ are polynomials in s.

The function $H(s)$ characterizes the continuous-time system and is called the *system function* or the *transfer function* of the system.

The roots of the transfer function denominator $A(s)$ are called the *poles* of the transfer function. We compute the poles from the equation

$$A(s) = \text{denominator}(H(s)) = 0$$

and designate by $s_{\mathrm{p}1}, s_{\mathrm{p}2}, s_{\mathrm{p}3}, \ldots$.

In a similar way, the roots of the transfer function numerator $B(s)$ are called the *zeros* of the transfer function. We obtain the zeros from the equation

$$B(s) = \text{numerator}(H(s)) = 0$$

and designate these zeros by $s_{\mathrm{z}1}, s_{\mathrm{z}2}, s_{\mathrm{z}3}, \ldots$.

The transfer function can be written in the factored form as

$$H(s) = H_0 \frac{\prod_k (s - s_{zk})}{\prod_i (s - s_{pi})} \tag{3.187}$$

The form (3.187) is known as the *pole-zero representation* of the transfer function. The real constant H_0 is called the *scale factor*.

Since the polynomials $A(s)$ and $B(s)$ have real coefficients, zeros and poles must be real or occur in complex conjugate pairs.

The simplest transfer function is the first-order transfer function

$$H(s) = \frac{b_1 s + b_0}{a_1 s + a_0}, \qquad a_1 \neq 0 \tag{3.188}$$

Systems described by (3.188) we call *first-order sections*.

Transfer functions of the form

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}, \qquad a_2 \neq 0 \tag{3.189}$$

are called *second-order transfer functions*, and they play an important role in analysis and design of continuous-time LTI systems. A system characterized by (3.189) we call a *biquadratic section* or *biquad*.

Any transfer function (3.186) can be expressed as a product of first-order (3.188) and second-order (3.189) transfer functions, which implies that any LTI system can be resolved into first-order sections and biquads.

For $s = j\omega$, $H(s)$ is the frequency response of the LTI system. Many properties of LTI systems can be closely associated with the characteristics of the system function in the s-plane, and in particular with the pole locations.

Partial Transfer Function and Transfer-Function Matrix. Consider a relaxed multiple-input multiple-output system with N inputs, $x_1, x_2, \ldots, x_k, \ldots, x_N$, and assume that the system has L outputs $y_1, y_2, \ldots, y_i, \ldots, y_L$. We define the *partial transfer function* between the ith output and the kth input to be the ratio of the output Laplace transform $Y_i(s)$ to the input Laplace transform $X_k(s)$, with the other inputs being identically zero:

$$H_{ik}(s) = \left. \frac{Y_i(s)}{X_k(s)} \right|_{x_1 = \cdots = x_{k-1} = x_{k+1} = \cdots = x_N = 0} \tag{3.190}$$

The input–output description of the system can be expressed in a matrix form:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_i(s) \\ \vdots \\ Y_L(s) \end{bmatrix} = \begin{bmatrix} H_{11}(s) & H_{12}(s) & \cdots & H_{1k}(s) & \cdots & H_{1N}(s) \\ H_{21}(s) & H_{22}(s) & \cdots & H_{2k}(s) & \cdots & H_{2N}(s) \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ H_{i1}(s) & H_{i2}(s) & \cdots & H_{ik}(s) & \cdots & H_{iN}(s) \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ H_{L1}(s) & H_{L2}(s) & \cdots & H_{Lk}(s) & \cdots & H_{LN}(s) \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_k(s) \\ \vdots \\ X_N(s) \end{bmatrix}$$

where the matrix

$$\mathbf{H}(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) & \cdots & H_{1k}(s) & \cdots & H_{1N}(s) \\ H_{21}(s) & H_{22}(s) & \cdots & H_{2k}(s) & \cdots & H_{2N}(s) \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ H_{i1}(s) & H_{i2}(s) & \cdots & H_{ik}(s) & \cdots & H_{iN}(s) \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ H_{L1}(s) & H_{L2}(s) & \cdots & H_{Lk}(s) & \cdots & H_{LN}(s) \end{bmatrix}$$

is called the *transfer-function matrix* of the system.

3.5 DISCRETE FOURIER TRANSFORM

Discrete Fourier transform (DFT) is one of the most important tools for signal processing techniques. Originally, it has been developed for calculating the Fourier coefficients and the Fourier transform on a digital computer.

3.5.1 Definition of the Discrete Fourier Transform

Consider a finite-length sequence

$$\{x_{(n)}\}_N = \{x_{(0)}, x_{(1)}, x_{(2)}, \ldots, x_{(n)}, \ldots, x_{(N-1)}\} \tag{3.191}$$

where the curly brackets are used to denote a sequence, and $x_{(n)}$ is the nth member of the sequence ($n = 0, 1, 2, \ldots, N-1$).

The *discrete Fourier transform* (DFT) is defined as another sequence of the same length:

$$\{X_{(k)}\}_N = \{X_{(0)}, X_{(1)}, X_{(2)}, \ldots, X_{(k)}, \ldots, X_{(N-1)}\} \tag{3.192}$$

with

$$X_{(k)} = \sum_{n=0}^{N-1} x_{(n)} w^{-kn} \tag{3.193}$$

and

$$w = \mathrm{e}^{j\frac{2\pi}{N}} \tag{3.194}$$

Note that (the asterisk denotes the complex conjugate)

$$w^* = w^{-1}$$

and

$$w^{l+mN} = w^l$$

for arbitrary integers l and m.

The *inverse discrete Fourier transform* (IDFT) is defined as

$$x_{(n)} = \frac{1}{N}\sum_{k=0}^{N-1} X_{(k)} w^{kn} \tag{3.195}$$

The two sequences, $\{x_{(n)}\}_N$ and $\{X_{(k)}\}_N$, are said to form a *discrete Fourier transform pair*, which is symbolized by

$$\{x_{(n)}\}_N \leftrightarrow \{X_{(k)}\}_N \tag{3.196}$$

or

$$\begin{aligned} \mathcal{D}\{x_{(n)}\}_N &= \{X_{(k)}\}_N \\ \mathcal{D}^{-1}\{X_{(k)}\}_N &= \{x_{(n)}\}_N \end{aligned} \tag{3.197}$$

Generally, the members of these sequences are complex numbers, and the discrete Fourier transform establishes a one-to-one correspondence between the two sequences.

The finite-length sequence $\{x_{(n)}\}_N$ can be periodically extended to form a periodic sequence of numbers, $\{p_{(n)}\}$, with period N:

$$\begin{aligned} p_{(n)} &= x_{(n)}, \quad (n = 0, 1, 2, \ldots, N-1) \\ p_{(n+mN)} &= p_{(n)}, \quad (m = \pm 1, \pm 2, \ldots) \end{aligned} \tag{3.198}$$

The discrete Fourier transform of the periodic sequence $\{p_{(n)}\}$ is defined as another periodic sequence, $\{P_{(k)}\}$, with the same period N:

$$\begin{aligned} P_{(k)} &= X_{(k)} \qquad (k = 0, 1, 2, \ldots, N-1) \\ P_{(k+mN)} &= P_{(k)} \qquad (m = \pm 1, \pm 2, \ldots) \end{aligned} \tag{3.199}$$

Often, the sequence $\{x_{(n)}\}_N$ is generated by sampling a continuous-time signal $x(t)$. We observe the signal over an interval of length T, and we take N equidistant samples at the points $t_0, t_0 + \Delta t, t_0 + 2\Delta t, \ldots, t_0 + (N-1)\Delta t$; frequently, $t_0 = 0$. The members of the sequence $\{x_{(n)}\}_N$ are

$$x_{(n)} = x(t_0 + n\Delta t)$$
$$n = 0, 1, 2, \ldots, N-1$$

The number of samples per unit interval of time defines the *sampling rate*

$$f_{\text{samp}} = \frac{1}{\Delta t}$$

also called the *sampling frequency*.

We associate a sequence of frequencies

$$\begin{aligned}\{f_{(k)}\}_N &= \{0, \Delta f, 2\Delta f, \ldots, k\Delta f, \ldots, (N-1)\Delta f\}\\ \Delta f &= \frac{1}{N} f_{\text{samp}}\end{aligned} \tag{3.200}$$

to the DFT sequence $\{X_{(k)}\}_N$; that is, the frequency

$$f_{(k)} = k\Delta f$$

corresponds to $X_{(k)}$. Sometimes, it is more convenient to introduce a sequence of angular frequencies:

$$\begin{aligned}\{\omega_{(k)}\}_N &= \{0, \Delta\omega, 2\Delta\omega, \ldots, k\Delta\omega, \ldots, (N-1)\Delta\omega\}\\ \Delta\omega &= 2\pi\Delta f = 2\pi\frac{1}{N} f_{\text{samp}}\end{aligned} \tag{3.201}$$

Definition equations of DFT and IDFT can be written in matrix form:

$$\begin{bmatrix} X_{(0)} \\ X_{(1)} \\ X_{(2)} \\ \vdots \\ X_{(N-1)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w^{-1} & w^{-2} & \cdots & w^{-(N-1)} \\ 1 & w^{-2} & w^{-4} & \cdots & w^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{-(N-1)} & w^{-2(N-1)} & \cdots & w^{-(N-1)^2} \end{bmatrix} \begin{bmatrix} x_{(0)} \\ x_{(1)} \\ x_{(2)} \\ \vdots \\ x_{(N-1)} \end{bmatrix}$$

or

$$\mathbf{X} = \mathbf{W}\,\mathbf{x}$$

and

$$\begin{bmatrix} x_{(0)} \\ x_{(1)} \\ x_{(2)} \\ \vdots \\ x_{(N-1)} \end{bmatrix} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w^{1} & w^{2} & \cdots & w^{(N-1)} \\ 1 & w^{2} & w^{4} & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{(N-1)} & w^{2(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix} \begin{bmatrix} X_{(0)} \\ X_{(1)} \\ X_{(2)} \\ \vdots \\ X_{(N-1)} \end{bmatrix}$$

or

$$\mathbf{x} = \frac{1}{N}\hat{\mathbf{W}}\,\mathbf{X}$$

where the matrix $\hat{\mathbf{W}}$ is obtained from $\mathbf{W}$ by changing w^{-k} to w^{k}.

Any efficient computational algorithms for the calculation of DFT and IDFT, which speeds up the evaluation of (3.193) and (3.195), is called a *fast Fourier transform* (FFT).

3.5.2 Properties of the Discrete Fourier Transform

Consider sequences $\{x_{(n)}\}$, $\{h_{(n)}\}$, and $\{y_{(n)}\}$ with transforms $\{X_{(k)}\}$, $\{H_{(k)}\}$, and $\{Y_{(k)}\}$, that is, $\{x_{(n)}\} \leftrightarrow \{X_{(k)}\}$, $\{h_{(n)}\} \leftrightarrow \{H_{(k)}\}$, and $\{y_{(n)}\} \leftrightarrow \{Y_{(k)}\}$. Unless otherwise stated, we assume that the sequences are either periodic or nonperiodic finite-length sequences.

We review several important properties relevant for the theory of discrete-time signals and systems.

Uniqueness. The discrete Fourier transform is *unique*:

$$\{x_{(n)}\} = \{y_{(n)}\} \Leftrightarrow \{X_{(k)}\} = \{Y_{(k)}\} \tag{3.202}$$

Homogeneity. The operators $\mathcal{D}$ and $\mathcal{D}^{-1}$ are *homogeneous*:

$$\begin{aligned} \mathcal{D}(K\{x_{(n)}\}) &= K\{X_{(k)}\} \\ \mathcal{D}^{-1}(K\{X_{(k)}\}) &= K\{x_{(n)}\} \end{aligned} \tag{3.203}$$

for any real or complex constant K.

Remark 3.1. The notation $K\{x_{(n)}\}$ symbolizes multiplication of a sequence by a constant; it means that each member of the sequence $\{x_{(n)}\}$ is multiplied by the constant K.

Additivity. The operators $\mathcal{D}$ and $\mathcal{D}^{-1}$ are *additive*:

$$\begin{aligned} \mathcal{D}(\{x_{(n)}\} + \{y_{(n)}\}) &= \{X_{(k)}\} + \{Y_{(k)}\} \\ \mathcal{D}^{-1}(\{X_{(k)}\} + \{Y_{(k)}\}) &= \{x_{(n)}\} + \{y_{(n)}\} \end{aligned} \tag{3.204}$$

The two sequences can be periodic with the same period N, or they can be nonperiodic finite-length sequences of the same length N.

Remark 3.2. The notation $\{x_{(n)}\}+\{y_{(n)}\}$ symbolizes addition of two sequences element by element.

Linearity. The homogeneity and additivity properties can be combined into the *linearity* property as follows:

$$\begin{aligned} \mathcal{D}(A\{x_{(n)}\} + B\{y_{(n)}\}) &= A\{X_{(k)}\} + B\{Y_{(k)}\} \\ \mathcal{D}^{-1}(A\{X_{(k)}\} + B\{Y_{(k)}\}) &= A\{x_{(n)}\} + B\{y_{(n)}\} \end{aligned} \tag{3.205}$$

for arbitrary real or complex constants A and B.

Shifting. The operation of shifting a periodic sequence maps into the operation of multiplication its transform by a constant:

$$\begin{aligned} \mathcal{D}\{x_{(n-\mu)}\} &= \{w^{-\mu k} X_{(k)}\} \\ \mathcal{D}^{-1}\{w^{-\mu k} X_{(k)}\} &= \{x_{(n-\mu)}\} \end{aligned} \tag{3.206}$$

for an arbitrary integer μ, and

$$w = \mathrm{e}^{j\frac{2\pi}{N}}$$

with N being the period of the sequence.

Cyclic Convolution. *Cyclic convolution* of two periodic sequences with period N is defined as

$$\{y_{(n)}\} = \{x_{(n)}\} * \{h_{(n)}\}$$
$$y_{(n)} = \sum_{\mu=0}^{N-1} x_{(\mu)} h_{(n-\mu)} \tag{3.207}$$

and is a periodic sequence with period N. The cyclic convolution is a commutative operation.

DFT of the cyclic convolution of two periodic sequences is a periodic sequence whose members are products of the corresponding members of the individual DFTs:

$$\mathcal{D}\{x_{(n)}\} * \{h_{(n)}\} = \mathcal{D}\{h_{(n)}\} * \{x_{(n)}\} = \{X_{(k)} H_{(k)}\}$$
$$\mathcal{D}^{-1}\{X_{(k)} H_{(k)}\} = \{x_{(n)}\} * \{h_{(n)}\} = \{h_{(n)}\} * \{x_{(n)}\}$$

Symmetry. For any periodic sequence we have

$$\{x^*_{(-n)}\} \leftrightarrow \{X^*_{(k)}\}$$
$$\{x_{(-n)}\} \leftrightarrow \{X_{(-k)}\} \tag{3.208}$$

If a periodic sequence is real, it follows that

$$X_{(-k)} = X^*_{(k)}$$

showing that we effectively need to know only $X_{(k)}$ for $k = 0, 1, 2, \ldots, \frac{N}{2}$.

Parseval's Identity and Power Spectrum. For any periodic sequence with period N we have

$$\sum_{n=0}^{N-1} |x_{(n)}|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X_{(k)}|^2 \tag{3.209}$$

which is known as the *Parseval's identity*.

The sequence of numbers $\{\frac{1}{N}|X_{(k)}|^2\}$ is called the *power spectral density*, and the plot of $\frac{1}{N}|X_{(k)}|^2$ against the integer k is the *power spectrum* of the sequence ($k = 0, 1, 2, \ldots, N-1$).

Cyclic Correlation. *Cyclic correlation* of two periodic sequences with period N is defined as

$$\{R_{xy(n)}\} = \{x_{(n)}\} \star \{y_{(n)}\}$$
$$R_{xy(n)} = \frac{1}{N} \sum_{\mu=0}^{N-1} x_{(\mu)} y_{(n+\mu)} \tag{3.210}$$

and is a periodic sequence with period N.

DFT of the cyclic correlation is

$$\mathcal{D}\{x_{(n)}\} \star \{y_{(n)}\} = \left\{\frac{1}{N} X^*_{(k)} Y_{(k)}\right\}$$

Cyclic autocorrelation of a periodic sequence is $\{R_{xx(n)}\}$, and it forms a DFT pair with the power spectral density

$$\{R_{xx(n)}\} \leftrightarrow \left\{\frac{1}{N}|X_{(k)}|^2\right\} \tag{3.211}$$

3.5.3 Computation of the Fourier Series Coefficients by DFT

Consider a real continuous-time periodic signal $x(t)$, with period T, that can be represented by the Fourier series

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{jk\omega_1 t} \tag{3.212}$$

with the coefficients

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_1 t}\, dt \tag{3.213}$$

and the fundamental angular frequency

$$\omega_1 = \frac{2\pi}{T} \tag{3.214}$$

Also, we know that for real signals

$$C_{-k} = C_k^*$$

The integral in (3.213) which determines the Fourier coefficients C_k can be evaluated by using numerical techniques; therefore, the integral must be approximated by a summation:

$$C_k = \frac{1}{T}\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} x(t) e^{-jk\omega_1 t}\, dt$$

$$C_k \approx \frac{1}{T}\sum_{n=0}^{N-1} x(n\Delta t) e^{-jk\omega_1 n\Delta t}\, \Delta t$$

where we choose the number of subintervals N and compute the time step

$$\Delta t = \frac{T}{N}$$

After some simplification we may write

$$C_k \approx \frac{1}{N}\sum_{n=0}^{N-1} x_{(n)} w^{-kn} = \frac{1}{N} X_{(k)} \tag{3.215}$$

where

$$x_{(n)} = x(n\Delta t), \qquad w = \mathrm{e}^{j\frac{2\pi}{N}}, \quad \{X_{(k)}\} = \mathcal{D}\{x_{(n)}\}$$

The summation in (3.215) gives only N different coefficients because w is periodic with period N. It means that the approximation (3.215) will be accurate only if we can accurately represent the signal $x(t)$ by a truncated series

$$x(t) = \sum_{k=-K}^{K} C_k \, \mathrm{e}^{jk\omega_1 t}$$

which implies that either we assume $C_k = 0$, for $|k| > K$, or we neglect the Fourier coefficients for $|k| > K$.

According to the sampling theorem the maximum spectral frequency, $f_{\max} = \frac{1}{2\pi} K\omega_1 = K\frac{1}{T}$, and the sampling frequency, $f_{\text{samp}} = \frac{1}{\Delta t} = N\frac{1}{T}$, must satisfy the condition $f_{\max} < 2 f_{\text{samp}}$, which yields

$$K < 2N$$

We conclude that the Fourier series coefficients, C_k, of a real periodic function, $x(t)$, can be computed by DFT if we may assume that

$$C_k = \begin{cases} \dfrac{1}{N} X_{(k)}, & |k| < \dfrac{N}{2} \\ 0, & |k| \geq \dfrac{N}{2} \end{cases}$$

and if the signal is uniformly sampled over one period. In other words, we expect that the signal has negligible spectral components at frequencies higher than $\frac{N}{2T}$.

3.5.4 Computation of the Fourier Integral by DFT

Consider a real nonperiodic signal $x(t)$ that can be represented by the Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) \mathrm{e}^{j\omega t} \, \mathrm{d}\omega$$

where

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) \mathrm{e}^{-j\omega t} \, \mathrm{d}t \tag{3.216}$$

For real signals

$$X(-j\omega) = X^*(j\omega)$$

Without lack of generality, assume that the origin of the time axis has been adjusted so that the integral (3.216) can be approximated by one with finite limits

$$X(j\omega) \approx \int_0^T x(t)\mathrm{e}^{-j\omega t}\,\mathrm{d}t$$

and evaluated by the summation

$$X(j\omega) \approx \sum_{n=0}^{N-1} x_{(n)}\mathrm{e}^{-\mathrm{j}\omega n\Delta t}\,\Delta t$$

where

$$\Delta t = \frac{T}{N}, \qquad x_{(n)} = x(n\Delta t)$$

The number of samples, N, is chosen to ensure accurate numerical integration.

Our goal is to compute the Fourier transform of $x(t)$, in terms of its samples, by means of DFT. We sample the signal at the rate $f_{\text{samp}} = \frac{1}{\Delta t}$, and we assume that the transform $X(j\omega)$ can be neglected for $|\omega| \geq \omega_{\max} = \frac{1}{2}2\pi f_{\text{samp}}$; next, we want the signal samples to be accurately represented by the finite-limits integral

$$x(n\Delta t) = \frac{1}{2\pi}\int_{-\omega_{\max}}^{+\omega_{\max}} X(\mathrm{j}\omega)\mathrm{e}^{j\omega n\Delta t}\,\mathrm{d}\omega$$

DFT can be used to determine only N values of $X(j\omega)$ from N values (samples) of $x(t)$. Therefore, we compute

$$X(jk\Delta\omega) = \Delta t\sum_{n=0}^{N-1} x_{(n)}\mathrm{e}^{-jk\Delta\omega n\Delta t}$$

with

$$\Delta\omega = \frac{1}{N}2\pi\frac{1}{\Delta t}$$

because DFT and IDFT must be periodic with period N. We obtain

$$X(jk\Delta\omega) = \Delta t\sum_{n=0}^{N-1} x_{(n)}w^{-kn} = \frac{1}{f_{\text{samp}}}X_{(k)}$$

where

$$w = \mathrm{e}^{j\frac{2\pi}{N}}, \qquad \{X_{(k)}\} = \mathcal{D}\{x_{(n)}\}$$

We conclude that the Fourier transform, $X(j\omega)$, of a real nonperiodic function, $x(t)$, can be computed by DFT if we may assume that

$$X(jk\Delta\omega) = \begin{cases} \dfrac{1}{f_{\text{samp}}} X_{(k)}, & |k| < \dfrac{N}{2} \\ 0, & |k| \geq \dfrac{N}{2} \end{cases}$$

and if the signal is uniformly sampled at the rate $f_{\text{samp}} = \frac{1}{\Delta t} = \frac{T}{N}$ over a time interval of duration T. In other words, we expect that the signal has negligible spectral components at frequencies higher than $\frac{N}{2T}$.

3.5.5 Frequency Response of Discrete-Time Systems

Consider a relaxed, single-input, single-output, discrete-time LTI system described by means of a linear constant-coefficients difference equation

$$\sum_{m=0}^{M} a_m y_{(n-m)} = \sum_{l=0}^{L} b_l x_{(n-l)}$$

(We drop the curly braces around $y_{(n-m)}$ and $x_{(n-l)}$ for the case of simplicity.)

Assume that the system is excited by an input sequence $\{x_{(n)}\}$ of length N, and observe the output sequence $\{y_{(n)}\}$ of the same length N. By applying the N-point DFT to both sides of the difference equation, and after employing the linearity property and the shifting property, we obtain an equation relating the discrete Fourier transforms of the two sequences

$$\left(\sum_{m=0}^{M} a_m w^{-mk}\right) Y_{(k)} = \left(\sum_{l=0}^{L} b_l w^{-lk}\right) X_{(k)}$$

where $w = \mathrm{e}^{(j2\pi/N)}$, and $k = 0, 1, 2, \ldots, N$. The system transforms the input sequence by multiplying each member of the input DFT with the factor

$$H_{(k)} = \frac{\displaystyle\sum_{l=0}^{L} b_l w^{-lk}}{\displaystyle\sum_{m=0}^{M} a_m w^{-mk}}$$

The sequence $\{H_{(k)}\}$ can be generated from a rational function of complex variable

$$H(z) = \frac{N(z)}{D(z)}, \qquad N(z) = \sum_{l=0}^{L} b_l z^{-l}, \qquad D(z) = \sum_{m=0}^{M} a_m z^{-m}$$

if we make the substitution

$$z = \mathrm{e}^{j\theta}$$

with

$$\theta = 2\pi \frac{k}{N}$$

The function $H(z)$ characterizes the discrete-time system and is called the *system function* or the *transfer function* of the system. The quantity θ is known as the *digital angular frequency*. In this book the quantity $\frac{\theta}{2\pi}$ is referred to as the *digital frequency*. The complex function $H(e^{j\theta})$ in terms of real variable θ is called the *frequency response* of the system.

3.6 THE *z* TRANSFORM

Linear time-invariant discrete-time systems, characterized by linear constant-coefficient difference equations, are efficiently analyzed by using the z transform that transforms the difference equations into algebraic equations which are easier to manipulate.

3.6.1 Definition of the z Transform

The *two-sided z transform* (or bilateral z transform) of a sequence $\{x_{(n)}\}$ is defined as

$$X_b(z) = \mathcal{Z}_b\{x_{(n)}\} = \sum_{n=-\infty}^{\infty} x_{(n)} z^{-n} \tag{3.217}$$

for all z for which $X_b(z)$ converges.

The *one-sided z transform* (or unilateral z transform) of a sequence $\{x_{(n)}\}$ is defined as

$$X(z) = \mathcal{Z}\{x_{(n)}\} = \sum_{n=0}^{\infty} x_{(n)} z^{-n} \tag{3.218}$$

for all z for which $X(z)$ converges.

We shall be concerned with z transforms whose singularities are poles. Mostly, we shall consider causal sequences whose members are zero for $n < 0$. Unless otherwise stated, in this book the term *z transform* refers to the one-sided z transform.

The region of convergence (ROC) for $X(z)$ is an annular ring in the z plane

$$R_{\min} < |z| < R_{\max}$$

and (unless obvious) must be included in the specification of $X(z)$ in order for the z transform to be complete.

Sequence $\{x_{(n)}\}$ is said to be the *inverse z transform* of $X(z)$ and can be uniquely determined by

$$x_{(n)} = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1}\, dz \tag{3.219}$$

where Γ is a contour in the counterclockwise sense enclosing all the singularities of $X(z)$.

The sequence $\{x_{(n)}\}$ and the complex function $X(z)$ are said to form a *z transform pair*, which is symbolized by

$$\{x_{(n)}\} \leftrightarrow X(z) \tag{3.220}$$

or

$$\begin{aligned} \mathcal{Z}\{x_{(n)}\} &= X(z) \\ \mathcal{Z}^{-1}X(z) &= \{x_{(n)}\} \end{aligned} \tag{3.221}$$

3.6.2 Properties of the z Transform

Consider sequences $\{x_{(n)}\}$, $\{h_{(n)}\}$, and $\{y_{(n)}\}$ with transforms $X(z)$, $H(z)$, and $Y(z)$, that is, $\{x_{(n)}\} \leftrightarrow X(z)$, $\{h_{(n)}\} \leftrightarrow H(z)$, and $\{y_{(n)}\} \leftrightarrow Y(z)$. We review several important properties relevant for the theory of discrete-time signals and systems.

Uniqueness. The z transform is *unique*:

$$\{x_{(n)}\} = \{y_{(n)}\} \Leftrightarrow X(z) = Y(z) \tag{3.222}$$

Homogeneity. The operators $\mathcal{Z}$ and $\mathcal{Z}^{-1}$ are *homogeneous*:

$$\begin{aligned} \mathcal{Z}(K\{x_{(n)}\}) &= KX(z) \\ \mathcal{Z}^{-1}(KX(z)) &= K\{x_{(n)}\} \end{aligned} \tag{3.223}$$

for any real or complex constant K.

Remark 3.3. The notation $K\{x_{(n)}\}$ symbolizes multiplication of a sequence by a constant; it means that each member of the sequence $\{x_{(n)}\}$ is multiplied by the constant K.

Additivity. The operators $\mathcal{Z}$ and $\mathcal{Z}^{-1}$ are *additive*:

$$\begin{aligned} \mathcal{Z}(\{x_{(n)}\} + \{y_{(n)}\}) &= X(z) + Y(z) \\ \mathcal{Z}^{-1}(X(z) + Y(z)) &= \{x_{(n)}\} + \{y_{(n)}\} \end{aligned} \tag{3.224}$$

Remark 3.4. The notation $\{x_{(n)}\}+\{y_{(n)}\}$ symbolizes addition of two sequences element by element.

Linearity. The homogeneity and additivity properties can be combined into the *linearity* property as follows:

$$\begin{aligned} \mathcal{Z}(A\{x_{(n)}\} + B\{y_{(n)}\}) &= AX(z) + BY(z) \\ \mathcal{Z}^{-1}(AX(z) + BY(z)) &= A\{x_{(n)}\} + B\{y_{(n)}\} \end{aligned} \tag{3.225}$$

for arbitrary real or complex constants A and B.

Shifting. The operation of *shifting* (*translation*) a causal sequence maps into the operation of multiplication of its z transform:

$$
\begin{aligned}
\mathcal{Z}\{x_{(n-\mu)}\} &= z^{-\mu}X(z) \\
\mathcal{Z}^{-1}z^{-\mu}X(z) &= \{x_{(n-\mu)}\} \\
\mathcal{Z}\{x_{(n+\mu)}\} &= z^{\mu}X(z) - z^{\mu}\sum_{k=0}^{\mu-1} x_{(k)}z^{-k}
\end{aligned}
\tag{3.226}
$$

for an arbitrary positive integer μ. Since a negative shift (delay) of $\mu = 1$ causes $X(z)$ to be multiplied by z^{-1}, z^{-1} is referred to as the *unit delay operator*.

Convolution. *Convolution* of two causal sequences is defined as

$$
\begin{aligned}
\{y_{(n)}\} &= \{x_{(n)}\} * \{h_{(n)}\} \\
y_{(n)} &= \sum_{\mu=0}^{\infty} x_{(\mu)}h_{(n-\mu)}
\end{aligned}
\tag{3.227}
$$

The convolution is a commutative operation.

The z transform of the convolution of two causal sequences is a product of the z transforms of the sequences

$$
\begin{aligned}
\mathcal{Z}\{x_{(n)}\} * \{h_{(n)}\} &= \mathcal{Z}\{h_{(n)}\} * \{x_{(n)}\} = X(z)H(z) \\
\mathcal{Z}^{-1}(X(z)H(z)) &= \{x_{(n)}\} * \{h_{(n)}\} = \{h_{(n)}\} * \{x_{(n)}\}
\end{aligned}
$$

Scaling. Multiplying each member of a sequence by w^{-n} maps into the z transform of the scaled argument:

$$
\begin{aligned}
\mathcal{Z}(w^{-n}\{x_{(n)}\}) &= X(wz) \\
\mathcal{Z}^{-1}X(wz) &= w^{-n}\{x_{(n)}\}
\end{aligned}
\tag{3.228}
$$

Differentiation.

$$
\begin{aligned}
\mathcal{Z}(n\{x_{(n)}\}) &= -z\frac{\mathrm{d}X(z)}{\mathrm{d}z} \\
\mathcal{Z}^{-1}z\frac{\mathrm{d}X(z)}{\mathrm{d}z} &= -n\{x_{(n)}\}
\end{aligned}
\tag{3.229}
$$

Standard z Transform Pairs. Some useful z transform pairs for some real causal sequences $\{x_{(n)}\}$ are as follows:

$$
\begin{array}{lcl}
x_{(n)} & \leftrightarrow & X(z) \\
\delta_{(n)} & \leftrightarrow & 1 \\
u_{(n)} & \leftrightarrow & \dfrac{z}{z-1} = \dfrac{1}{1-z^{-1}} \\
a^n u_{(n)} & \leftrightarrow & \dfrac{z}{z-a} = \dfrac{1}{1-az^{-1}} \\
n\, u_{(n)} & \leftrightarrow & \dfrac{z}{(z-1)^2} = \dfrac{z^{-1}}{(1-z^{-1})^2} \\
\sin(\theta n)\, u_{(n)} & \leftrightarrow & \dfrac{z\sin(\theta)}{z^2 - 2z\cos(\theta) + 1} \\
\cos(\theta n)\, u_{(n)} & \leftrightarrow & \dfrac{z(z-\cos(\theta))}{z^2 - 2z\cos(\theta) + 1} \\
\mathrm{e}^{-\alpha n}\sin(\theta n)\, u_{(n)} & \leftrightarrow & \dfrac{z\,\mathrm{e}^{-\alpha}\sin(\theta)}{z^2 - 2z\,\mathrm{e}^{-\alpha}\cos(\theta) + \mathrm{e}^{-2\alpha}} \\
\mathrm{e}^{-\alpha n}\cos(\theta n)\, u_{(n)} & \leftrightarrow & \dfrac{z(z-\mathrm{e}^{-\alpha}\cos(\theta))}{z^2 - 2z\,\mathrm{e}^{-\alpha}\cos(\theta) + \mathrm{e}^{-2\alpha}}
\end{array}
\tag{3.230}
$$

The pairs are generated by employing the definition of the transform or its properties.

3.6.3 Transfer Function of Discrete-Time Systems

Consider a relaxed, single-input, single-output, discrete-time LTI system described by means of a linear constant-coefficients difference equation

$$\sum_{m=0}^{M} a_m y_{(n-m)} = \sum_{l=0}^{L} b_l x_{(n-l)} \tag{3.231}$$

(We drop the curly braces around $y_{(n-m)}$ and $x_{(n-l)}$ for the case of simplicity.)

Assume that the system is excited by an input causal sequence $\{x_{(n)}\}$, and observe the output sequence $\{y_{(n)}\}$. By applying the z transform to both sides of the difference equation, and after employing the linearity property and the shifting property, we obtain an equation relating the z transforms of the two sequences:

$$\left(\sum_{m=0}^{M} a_m z^{-m}\right) Y(z) = \left(\sum_{l=0}^{L} b_l z^{-l}\right) X(z)$$

The system transforms the input sequence by multiplying the z transform of the input sequence with the factor

$$H(z) = \frac{\displaystyle\sum_{l=0}^{L} b_l z^{-l}}{\displaystyle\sum_{m=0}^{M} a_m z^{-m}} \tag{3.232}$$

The function $H(z)$ is a rational function in z^{-1}:

$$H(z) = \frac{N(z)}{D(z)}, \quad N(z) = \sum_{l=0}^{L} b_l z^{-l}, \quad D(z) = \sum_{m=0}^{M} a_m z^{-m}$$

which can, equivalently, be expressed as a rational function in z:

$$H(z) = \frac{B(z)}{A(z)}$$

where $A(z)$ and $B(z)$ are polynomials in z.

The function $H(z)$ characterizes the discrete-time system and is called the *system function* or the *transfer function* of the system. For causal systems the degree of the numerator polynomial, $B(z)$, is equal to or less than that of the denominator polynomial, $A(z)$.

The roots of the transfer function denominator $A(z)$ are called the *poles* of the transfer function. We compute the poles from the equation

$$A(z) = \text{denominator}(H(z)) = 0$$

and designate these poles by $z_{\text{p}1}, z_{\text{p}2}, z_{\text{p}3}, \ldots$.

In a similar way, the roots of the transfer function numerator $B(z)$ are called the *zeros* of the transfer function. We obtain the zeros from the equation

$$B(z) = \text{numerator}(H(z)) = 0$$

and designate these zeros by $z_{\text{z}1}, z_{\text{z}2}, z_{\text{z}3}, \ldots$.

The transfer function can be written in the factored form as

$$H(z) = H_0 \frac{\prod_k (z - z_{\text{z}k})}{\prod_i (z - z_{\text{p}i})} \tag{3.233}$$

The form (3.233) is known as the *pole-zero representation* of the transfer function. The real constant H_0 is called the *scale factor*.

Since the polynomials $A(z)$ and $B(z)$ have real coefficients, zeros and poles must be real or occur in complex-conjugate pairs.

The simplest transfer function is the first-order transfer function

$$H(z) = \frac{b_1 z + b_0}{a_1 z + a_0}, \quad a_1 \neq 0 \tag{3.234}$$

Systems described by (3.234) are called *first-order sections*.

Transfer functions of the form

$$H(z) = \frac{b_2 z^2 + b_1 z + b_0}{a_2 z^2 + a_1 z + a_0}, \quad a_2 \neq 0 \tag{3.235}$$

are called *second-order transfer functions*, and they play an important role in analysis and design of discrete-time LTI systems. A system characterized by (3.235) is called a *biquadratic section* or *biquad*.

Any transfer function (3.232) can be expressed as a product of first-order (3.234) and second-order (3.235) transfer functions, which implies that any discrete-time LTI system can be resolved into first-order sections and biquads.

Partial Transfer Function and Transfer-Function Matrix. Consider a relaxed multiple-input multiple-output system with N inputs, $x_1, x_2, \ldots, x_k, \ldots, x_N$, and assume that the system has L outputs $y_1, y_2, \ldots, y_i, \ldots, y_L$. We define the *partial transfer function* between the ith output and the kth input to be the ratio of the output z transform $Y_i(z)$ to the input z transform $X_k(z)$, with the other inputs being identically zero:

$$H_{ik}(z) = \left.\frac{Y_i(z)}{X_k(z)}\right|_{x_1=\cdots=x_{k-1}=x_{k+1}=\cdots=x_N=0} \tag{3.236}$$

The input–output description of the system can be expressed in a matrix form

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \\ \vdots \\ Y_i(z) \\ \vdots \\ Y_L(z) \end{bmatrix} = \begin{bmatrix} H_{11}(z) & H_{12}(z) & \cdots & H_{1k}(z) & \cdots & H_{1N}(z) \\ H_{21}(z) & H_{22}(z) & \cdots & H_{2k}(z) & \cdots & H_{2N}(z) \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ H_{i1}(z) & H_{i2}(z) & \cdots & H_{ik}(z) & \cdots & H_{iN}(z) \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ H_{L1}(z) & H_{L2}(z) & \cdots & H_{Lk}(z) & \cdots & H_{LN}(z) \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \\ \vdots \\ X_k(z) \\ \vdots \\ X_N(z) \end{bmatrix}$$

where the matrix

$$\mathbf{H}(z) = \begin{bmatrix} H_{11}(z) & H_{12}(z) & \cdots & H_{1k}(z) & \cdots & H_{1N}(z) \\ H_{21}(z) & H_{22}(z) & \cdots & H_{2k}(z) & \cdots & H_{2N}(z) \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ H_{i1}(z) & H_{i2}(z) & \cdots & H_{ik}(z) & \cdots & H_{iN}(z) \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ H_{L1}(z) & H_{L2}(z) & \cdots & H_{Lk}(z) & \cdots & H_{LN}(z) \end{bmatrix}$$

is called the *transfer-function matrix* of the system.

3.7 ANALYSIS OF LTI SYSTEMS BY TRANSFORM METHOD

Linear time-invariant (LTI) systems, characterized by linear constant-coefficient differential or difference equations, are efficiently analyzed by using the Laplace transform or the z transform. The two transforms map the differential or difference equations into algebraic equations which are easier to manipulate. This section illustrates step-by-step procedures for analyzing LTI systems in the transform domain. For the given block diagram of a system the required equations are formulated and mapped by a suitable transform into a system of algebraic equations. The set of algebraic equations is solved to find the system response in the transform domain. Next, by the inverse transform, the system response is computed as a continuous-time function or sequence [19–22]

3.7.1 Continuous-Time LTI Systems

Consider a continuous-time linear time-invariant (CTLTI) system specified by its block diagram, as shown in Fig. 3.1, and assume zero initial conditions—that is, that the system is at rest. Our target is to find the response of the system—that is, the signals at all nodes—for an excitation applied at node 1. To make the analysis simple we proceed as follows:

1. Label all nodes by consecutive integer numbers starting with 1.
2. Assume that the signals at the nodes are y_1, y_2, and so on.
3. Assume that the excitation is a known function of time, $x(t)$.
4. Write the equations characterizing each block of the diagram. Number of equations equals the number of blocks and equals the number of nodes.
5. Apply the Laplace transform to both sides of each equation.
6. Solve the set of algebraic equations, obtained as a result of the Laplace transform, to find the transforms of the signals at the nodes, that is, compute $Y_1(s)$, $Y_2(s)$, in terms of $X(s)$ and system parameters.
7. Find the required transfer functions by dividing the transform at a node, $Y_i(s)$, by the transform of the excitation, $X(s)$; assume zero initial conditions.
8. Find the inverse Laplace transform of $Y_1(s)$, $Y_2(s)$, ..., to obtain the response $y_1(t)$, $y_2(t)$, ..., of the system.

We analyze the system depicted in Fig. 3.1 as follows:

1. There are six nodes that we label 1, 2, 3, 4, 5, and 6.
2. The signals at the nodes are designated by y_1, y_2, y_3, y_4, y_5, and y_6.
3. The excitation is a known given function of time, $x(t)$.

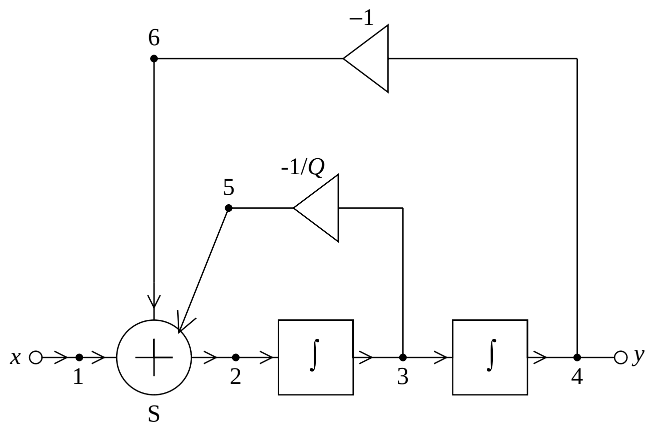

Figure 3.1 A continuous-time LTI system.

4. The equations characterizing each block of the diagram are

$$
\begin{aligned}
y_1 &= x(t) \\
y_2 &= y_1 + y_5 + y_6 \\
y_3 &= \omega \textstyle\int_0^t y_2(\tau)\, \mathrm{d}\tau \\
y_4 &= \omega \textstyle\int_0^t y_3(\tau)\, \mathrm{d}\tau \\
y_5 &= -\frac{1}{Q} y_3 \\
y_6 &= -y_4
\end{aligned}
\tag{3.237}
$$

The initial conditions are assumed to be zero. Notice that we have six blocks (excitation, adder, two integrators, two amplifiers), six nodes, and six equations.

5. We apply the Laplace transform to both sides of each equation in (3.237). We employ the four properties of the Laplace transform (uniqueness, homogeneity, additivity, differentiating) and obtain

$$
\begin{aligned}
Y_1(s) &= X(s) \\
Y_2(s) &= Y_1(s) + Y_5(s) + Y_6(s) \\
Y_3(s) &= \omega \frac{Y_2(s)}{s} \\
Y_4(s) &= \omega \frac{Y_3(s)}{s} \\
Y_5(s) &= -\frac{1}{Q} Y_3(s) \\
Y_6(s) &= -Y_4(s)
\end{aligned}
\tag{3.238}
$$

6. Solution of the set of algebraic equations (3.238) yields the complex response $Y_1(s), \ldots, Y_6(s)$ in terms of the complex excitation $X(s)$ and the system parameters Q and ω:

$$
\begin{aligned}
Y_1(s) &= X(s) \\
Y_2(s) &= \frac{s^2}{s^2 + \dfrac{\omega}{Q}s + \omega^2} X(s) \\
Y_3(s) &= \frac{\omega s}{s^2 + \dfrac{\omega}{Q}s + \omega^2} X(s) \\
Y_4(s) &= \frac{\omega^2}{s^2 + \dfrac{\omega}{Q}s + \omega^2} X(s) \\
&\ldots
\end{aligned}
$$

7. Assume that we want to compute the transfer functions $H_i(s) = Y_i(s)/X(s)$, $(i = 2, 3, 4)$. We find

$$
\begin{aligned}
H_2(s) &= \frac{Y_2(s)}{X(s)} = \frac{s^2}{s^2 + \dfrac{\omega}{Q}s + \omega^2} \\
H_3(s) &= \frac{Y_3(s)}{X(s)} = \frac{\omega s}{s^2 + \dfrac{\omega}{Q}s + \omega^2} \\
H_4(s) &= \frac{Y_4(s)}{X(s)} = \frac{\omega^2}{s^2 + \dfrac{\omega}{Q}s + \omega^2}
\end{aligned}
$$

8. If required, we can find the inverse Laplace transform of $Y_1(s)$, $Y_2(s)$, $\ldots$ to obtain the response $y_1(t)$, $y_2(t)$, $\ldots$ for the given excitation; for example, the step response of the system at node 3, for $Q = \frac{3}{2}$, $\omega = 1$, is

$$
y_3(t) = \mathcal{L}^{-1}(H_3(s)X(s)) = \mathcal{L}^{-1}\frac{H_3(s)}{s} = \frac{3}{2\sqrt{2}} e^{\frac{-1}{3}t} \sin\left(\frac{2\sqrt{2}}{3}t\right) u(t)
$$

Notice that we have assumed the zero initial conditions. Nonzero initial conditions would change the equations

$$
\begin{aligned}
y_3 &= y_3(0^-) + \omega \int_0^t y_2(\tau)\,\mathrm{d}\tau \\
y_4 &= y_4(0^-) + \omega \int_0^t y_3(\tau)\,\mathrm{d}\tau
\end{aligned}
$$

and the algebraic equation

$$
\begin{aligned}
Y_3(s) &= \frac{y_3(0^-)}{s} + \omega\frac{Y_2(s)}{s} \\
Y_4(s) &= \frac{y_4(0^-)}{s} + \omega\frac{Y_3(s)}{s}
\end{aligned}
$$

By definition, when computing a transfer function, the initial conditions must be taken to be zero.

3.7.2 Discrete-Time LTI Systems

Consider a discrete-time linear time-invariant (DTLTI) system specified by its block diagram, as shown in Fig. 3.2, and assume zero initial conditions—that is, that the system is at rest. Our target is to find the response of the system—that is, the sequences at all nodes—for an excitation applied at node 1. To make the analysis simple we proceed as follows:

1. Label all nodes by consecutive integer numbers starting with 1.
2. Assume that the sequences at the nodes are $\{y_{1(n)}\}$, $\{y_{2(n)}\}$, and so on.
3. Assume that the excitation is a known sequence, $\{x_{(n)}\}$.
4. Write the equations characterizing each block of the diagram. Number of equations equals the number of blocks and equals the number of nodes.
5. Apply the z transform to both sides of each equation.
6. Solve the set of algebraic equations, obtained as a result of the z transform, to find the transforms of the sequences at the nodes—that is, compute $Y_1(z)$, $Y_2(z)$, ..., in terms of $X(z)$ and system parameters.
7. Find the required transfer functions by dividing the transform at a node, $Y_i(z)$, by the transform of the excitation, $X(z)$; assume zero initial conditions.
8. Find the inverse z transform of $Y_1(z)$, $Y_2(z)$, ... to obtain the response $\{y_{1(n)}\}$, $\{y_{2(n)}\}$, ... of the system.

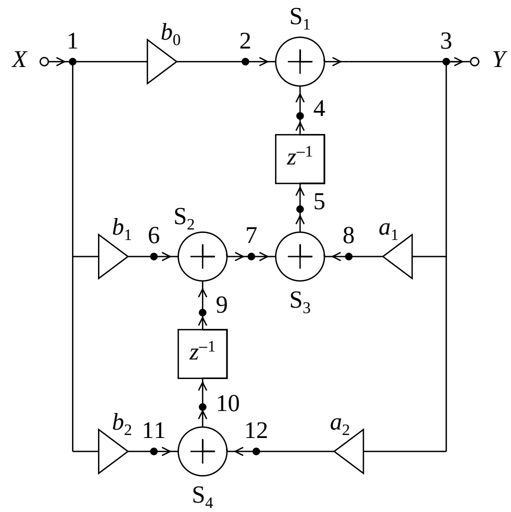

Figure 3.2 Discrete-time LTI system.

We analyze the system depicted in Fig. 3.2 as follows:

1. There are 12 nodes that we label $1, 2, 3, \ldots, 12$.
2. We designate the signals at the nodes by $y_1(n)$, $y_2(n)$, $y_3(n)$, $\ldots$, $y_{12}(n)$.
3. The excitation is a known given sequence, $x(n)$.
4. The equations characterizing each block of the diagram are

$$\begin{aligned}
y_1(n) &= x(n) \\
y_2(n) &= b_0 y_1(n) + x_{b0}(n) \\
y_3(n) &= y_2(n) + y_4(n) \\
y_4(n) &= y_5(n-1) \\
y_5(n) &= y_7(n) + y_8(n) \\
y_6(n) &= b_1 y_1(n) + x_{b1}(n) \\
y_7(n) &= y_6(n) + y_9(n) \\
y_8(n) &= a_1 y_3(n) + x_{a1}(n) \\
y_9(n) &= y_{10}(n-1) \\
y_{10}(n) &= y_{11}(n) + y_{12}(n) \\
y_{11}(n) &= b_2 y_1(n) + x_{b2}(n) \\
y_{12}(n) &= a_2 y_3(n) + x_{a2}(n)
\end{aligned} \tag{3.239}$$

The initial conditions are assumed to be zero. Notice that we have 12 blocks (excitation, 4 adders, 5 multipliers, 2 delays), 12 nodes, and 12 equations.

5. We apply the z transform to both sides of each equation in (3.239). We employ the four properties of the z transform (uniqueness, homogeneity, additivity, shifting) and obtain

$$\begin{aligned}
Y_1(z) &= X(z) \\
Y_2(z) &= b_0 Y_1(z) + X_{b0}(z) \\
Y_3(z) &= Y_2(z) + Y_4(z) \\
Y_4(z) &= z^{-1} Y_5(z) \\
Y_5(z) &= Y_7(z) + Y_8(z) \\
Y_6(z) &= b_1 Y_1(z) + X_{b1}(z) \\
Y_7(z) &= Y_6(z) + Y_9(z) \\
Y_8(z) &= a_1 Y_3(z) + X_{a1}(z) \\
Y_9(z) &= z^{-1} Y_{10}(z) \\
Y_{10}(z) &= Y_{11}(z) + Y_{12}(z) \\
Y_{11}(z) &= b_2 Y_1(z) + X_{b2}(z) \\
Y_{12}(z) &= a_2 Y_3(z) + X_{a2}(z)
\end{aligned} \tag{3.240}$$

6. Solution of the set of algebraic equations (3.240) yields the complex response $Y_1(z), \ldots, Y_{12}(z)$ in terms of the complex excitation $X(z)$, noise sources due to rounding the output of multipliers, $X_{a1}(z)$, $X_{a2}(z)$, $X_{b0}(z)$, $X_{b1}(z)$, $X_{b2}(z)$, and the multiplier coefficients a_1, a_2, b_0, b_1, b_2.

$$\begin{aligned}
Y_1(z) &= X(z) \\
&\vdots \\
Y_3(z) &= \cdots \\
&\vdots \\
Y_{12}(z) &= \cdots
\end{aligned}$$

Desired response, say y_3, in the z domain in terms of all excitations is

$$Y_3(z) = \frac{(b_0 + b_1 z^{-1} + b_2 z^{-2})X(z) + z^{-1}X_{a1}(z) + z^{-2}X_{a2}(z) + X_{b0}(z) + z^{-1}X_{b1}(z) + z^{-2}X_{b2}(z)}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

or, equivalently,

$$\begin{aligned} Y_3(z) = {} & \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} X(z) \\ & + \frac{z^{-1}}{1 - a_1 z^{-1} - a_2 z^{-2}} X_{a1}(z) \\ & + \frac{z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} X_{a2}(z) \\ & + \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2}} X_{b0}(z) \\ & + \frac{z^{-1}}{1 - a_1 z^{-1} - a_2 z^{-2}} X_{b1}(z) \\ & + \frac{z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} X_{b2}(z) \end{aligned}$$

7. Assume that we want to compute the transfer function of the system and the noise transfer functions for all multipliers. The transfer function of the system is defined by

$$H_3(z) = \left. \frac{Y_3(z)}{X(z)} \right|_{X_{a1}(z)=0, X_{a2}(z)=0, X_{b0}(z)=0, X_{b1}(z)=0, X_{b2}(z)=0}$$

and the noise transfer functions for each multiplier are defined by

$$\begin{aligned} H_{3a1}(z) &= \left. \frac{Y_3(z)}{X_{a1}(z)} \right|_{X(z)=0, X_{a2}(z)=0, X_{b0}(z)=0, X_{b1}(z)=0, X_{b2}(z)=0} \\ H_{3a2}(z) &= \left. \frac{Y_3(z)}{X_{a2}(z)} \right|_{X(z)=0, X_{a1}(z)=0, X_{b0}(z)=0, X_{b1}(z)=0, X_{b2}(z)=0} \\ H_{3b0}(z) &= \left. \frac{Y_3(z)}{X_{b0}(z)} \right|_{X(z)=0, X_{a1}(z)=0, X_{a2}(z)=0, X_{b1}(z)=0, X_{b2}(z)=0} \\ H_{3b1}(z) &= \left. \frac{Y_3(z)}{X_{b1}(z)} \right|_{X(z)=0, X_{a1}(z)=0, X_{a2}(z)=0, X_{b0}(z)=0, X_{b2}(z)=0} \\ H_{3b2}(z) &= \left. \frac{Y_3(z)}{X_{b2}(z)} \right|_{X(z)=0, X_{a1}(z)=0, X_{a2}(z)=0, X_{b0}(z)=0, X_{b1}(z)=0} \end{aligned}$$

We find the transfer function as

$$H_3(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

and find the noise transfer functions as

$$H_{3a1}(z) = \frac{z^{-1}}{1 - a_1z^{-1} - a_2z^{-2}}$$

$$H_{3a2}(z) = \frac{z^{-2}}{1 - a_1z^{-1} - a_2z^{-2}}$$

$$H_{3b0}(z) = \frac{1}{1 - a_1z^{-1} - a_2z^{-2}}$$

$$H_{3b1}(z) = \frac{z^{-1}}{1 - a_1z^{-1} - a_2z^{-2}}$$

$$H_{3b2}(z) = \frac{z^{-2}}{1 - a_1z^{-1} - a_2z^{-2}}$$

All transfer functions have the same denominator $1 - a_1z^{-1} - a_2z^{-2}$.

8. If required, we can find the inverse z transform of $Y_1(z)$, $Y_2(z)$, ... to obtain the response $y_1(n)$, $y_2(n)$, ... for the given excitation; for example, the impulse response of the system at node 3, for $a_1 = 0$, $a_2 = \frac{1}{2}$, $b_0 = 1$, $b_1 = 2$, and $b_2 = 1$, is

$$y_3(n) = Z^{-1}(H_3(z)X(z)) = Z^{-1}(H_3(z))$$
$$= 2^{-n/2}\left(-\cos(n\frac{\pi}{2}) + \sqrt{8}\sin(n\frac{\pi}{2})\right)u(n) + 2\delta(n)$$

and is shown in Fig. 3.3.

Multipliers of discrete-time systems are single-input single-output blocks defined by the equation $y(n) = ax(n)$, where a is the multiplier coefficient, $y(n)$ is the multiplier output, and $x(n)$ is the multiplier input. When quantization to B bits is performed after each multiplication, an error occurs; this error is considered as a noise sequence superimposed on the signal at the output of the multiplier. Therefore, the multiplier with product quantization at the output is modeled by the equation

$$y(n) = ax(n) + x_a(n)$$

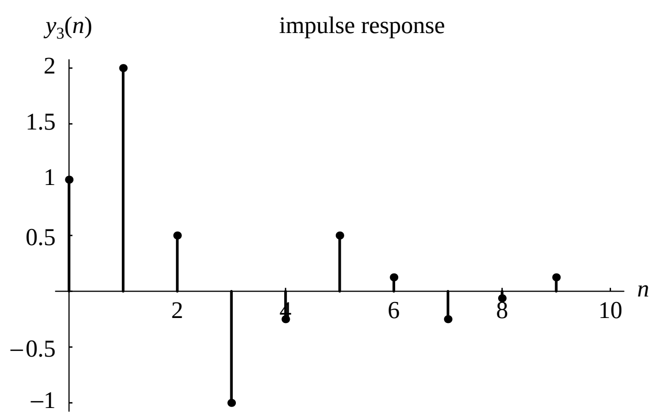

Figure 3.3 Impulse response of the discrete-time LTI system.

or in the z domain, taking the z transform of the left and right side of the above equation:

$$Y(z) = aX(z) + X_a(z)$$

The term $x_a(n)$ is the noise sequence due to product quantization superimposed on the signal at the output of the multiplier. Noise sequences are treated as inputs and are used for derivation of noise transfer functions. The noise transfer function $H_a(z)$, of a multiplier with the coefficient a, is defined as a ratio of the output z transform $Y(z)$ and the noise sequence z transform $X_a(z)$, assuming that all other inputs are set to zero.

3.7.3 Analog LTI Circuits

Consider a linear time-invariant analog circuit specified by its schematic, as shown in Fig. 3.4, and assume zero initial conditions—that is, that no energy exists in the circuit capacitors and inductors. Our target is to find the response of the circuit—that is, the currents and the voltages of the circuit branches. To make the analysis simple we proceed as follows:

1. Label all nodes by consecutive integer numbers starting from 0. The zero node is the ground node.
2. Assume that the node voltages are v_1, v_2, and so on. The voltage of the ground is zero, $v_0 = 0$.
3. Assume that the excitation (the voltage of an independent voltage source, or the current of an independent current source) is a known function of time, $x(t)$.
4. Write the Kirchhoff-current-law (KCL) equations for each node except node 0. Express the branch currents in terms of the node voltages. If the current of a branch cannot be expressed in terms of the node voltages, write the equation characterizing that branch, in terms of the node voltages and the current of that branch.
5. Apply the Laplace transform to both sides of each equation.

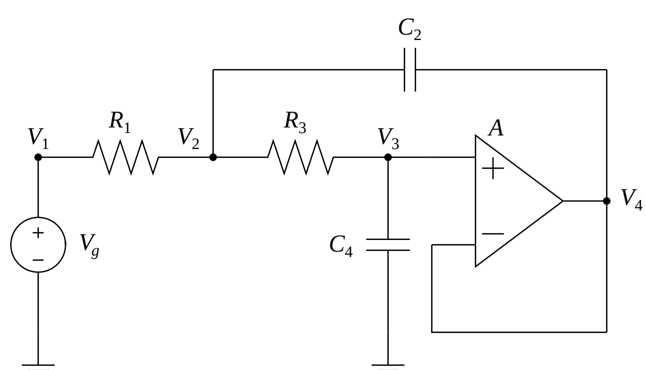

Figure 3.4 Analog LTI circuit.

6. Solve the set of algebraic equations, obtained as a result of the Laplace transform, to find the transforms of the circuit variables (that we call the *complex response*) in terms of the circuit parameters and the transform of the excitation.
7. Find the required transfer functions (also called *network functions*) by dividing the transform of a response by the transform of the excitation; assume zero initial conditions.
8. Find the inverse Laplace transform of the complex response to obtain the time-domain response of the circuit.

We analyze the analog LTI circuit depicted in Fig. 3.4 as follows:

1. There are five nodes that we label 0, 1, 2, 3, and 4.
2. We designate the node voltages by v_0, v_1, v_2, v_3, and v_4.
3. The excitation is a known given function of time, $v_g(t)$.
4. The circuit equations are

$$\begin{aligned} v_1 &= v_g(t) \\ \frac{v_2 - v_1}{R_1} + \frac{v_2 - v_3}{R_3} + C_2\mathrm{D}(v_2 - v_4) &= 0 \\ \frac{v_3 - v_2}{R_3} + C_4\mathrm{D}v_3 &= 0 \\ v_3 &= v_4 \end{aligned} \tag{3.241}$$

   The initial conditions are assumed to be zero. Notice that we have four equations (number of nodes minus one).

5. We apply the Laplace transform to both sides of each equation in (3.241). We employ the four properties of the Laplace transform (uniqueness, homogeneity, additivity, differentiating) and obtain

$$\begin{aligned} V_1(s) &= V_g(s) \\ \frac{V_2(s) - V_1(s)}{R_1} + \frac{V_2(s) - V_3(s)}{R_3} + C_2s(V_2(s) - V_4(s)) &= 0 \\ \frac{V_3(s) - V_2(s)}{R_3} + C_4sV_3(s) &= 0 \\ V_3(s) &= V_4(s) \end{aligned} \tag{3.242}$$

6. Solution of the set of algebraic equations (3.242) yields the complex response $V_1(s), \ldots, V_4(s)$ in terms of the complex excitation $V_g(s)$ and the circuit parameters R_1, R_3, C_2, and C_4

$$\begin{aligned}
V_1(s) &= V_g(s) \\
V_2(s) &= \frac{1 + C_4R_3s}{C_2C_4R_1R_3s^2 + C_4(R_1 + R_3)s + 1}V_g(s) \\
V_3(s) &= \frac{1}{C_2C_4R_1R_3s^2 + C_4(R_1 + R_3)s + 1}V_g(s) \\
V_4(s) &= \frac{1}{C_2C_4R_1R_3s^2 + C_4(R_1 + R_3)s + 1}V_g(s)
\end{aligned}$$

7. Assume that we want to compute the transfer functions $H_i(s) = V_i(s)/V_g(s)$ $(i = 1, 2, 3, 4)$. We find

$$\begin{aligned}
H_1(s) &= \frac{V_1(s)}{V_g(s)} = 1 \\
H_2(s) &= \frac{V_2(s)}{V_g(s)} = \frac{1 + C_4R_3s}{C_2C_4R_1R_3s^2 + C_4(R_1 + R_3)s + 1} \\
H_3(s) &= \frac{V_3(s)}{V_g(s)} = \frac{1}{C_2C_4R_1R_3s^2 + C_4(R_1 + R_3)s + 1} \\
H_4(s) &= \frac{V_4(s)}{V_g(s)} = \frac{1}{C_2C_4R_1R_3s^2 + C_4(R_1 + R_3)s + 1}
\end{aligned}$$

8. If required, we can find the inverse Laplace transform of $V_1(s)$, $V_2(s)$, ... to obtain the response $v_1(t)$, $v_2(t)$, ... for the given excitation; for example, the step response of the system at node 4, for $C_2 = 4C$, $C_4 = C$, and $R_3 = R_1 = R$, is

$$\begin{aligned}
v_4(t) &= \mathcal{L}^{-1}(H_4(s)V_g(s)) = \mathcal{L}^{-1}\frac{H_4(s)}{s} \\
&= \left(1 - e^{-t/(4CR)}\left(\cos(\frac{\sqrt{3}}{4CR}t) - \frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{4CR}t)\right)\right)u(t)
\end{aligned}$$

Figure 3.5 shows the step response for $C = 10$ nF and $R = 10$ kΩ.

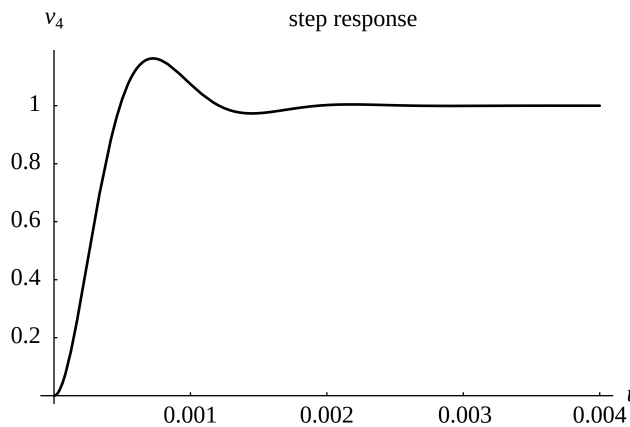

Figure 3.5 Step response of the analog LTI circuit.

PROBLEMS

3.1 Find the phasor representing the sinusoid

(a) $x_1(t) = 2\sin(\pi t)$,

(b) $x_2(t) = \sqrt{2}\sin(\frac{1}{2}\pi t + \frac{\pi}{6})$,

(c) $x_3(t) = 2\sqrt{2}\cos(3\pi t)$,

(d) $x_4(t) = 0.8\cos(0.25\pi t - \frac{\pi}{4})$,

(e) $x_5(t) = \frac{1}{2}\sin(\pi t/2) + \frac{1}{3}\cos(\pi t/2)$;

use the integral formula that establishes the phasor transformation.

3.2 Can you find the phasor for the following signal: $x(t) = \sqrt{2}\sin(125t) + 2\cos(250t)$? Explain the answer.

3.3 What signal quantities do we have to know when we perform the inverse phasor transformation? Find sinusoids corresponding to the phasors

(a) $X_1 = j\sqrt{2}$ at $f_1 = 1$ kHz,

(b) $X_2 = 2$ at $f_2 = 500$ Hz,

(c) $X_3 = 3\exp(j\pi/6)$ at $f_3 = 2$ kHz.

3.4 Use phasor transformation to find a sinusoidal signal, $x_s(t)$, equivalent to the signal

$$x(t) = \sin(2\pi t) + \frac{1}{2}\sin\left(2\pi t - \frac{\pi}{2}\right) + \frac{1}{3}\sin\left(2\pi t - \frac{\pi}{3}\right) + \frac{1}{4}\sin\left(2\pi t - \frac{\pi}{4}\right)$$

3.5 A system is defined by the input–output relation

$$\mathrm{D}^2 y(t) + a_1 \mathrm{D} y(t) + a_0 y(t) = b_0 x(t)$$

and excited by a sinusoidal signal $x(t) = X_m \cos(\omega t + \xi)$. Determine the steady-state response. Assume $a_1 = 2$, $a_0 = 4$, $b_0 = 1$, $X_m = \sqrt{2}$, $\omega = 2$, $\xi = 0$.

3.6 Compute and plot the steady-state response of a system specified by

$$D^2y(t) + \frac{\omega}{Q}Dy(t) + \omega^2 y(t) = \frac{\omega}{Q}x(t)$$

and excited by $x(t) = X_m\cos(\omega t) + X_m\cos(2\omega t)$, for $\omega = 2$, $Q = 4$, $X_m = 1$.

3.7 Find the transfer function (frequency response) of a system described by

$$a_2D^2y(t) + a_1Dy(t) + a_0y(t) = b_2D^2x(t) + b_1Dx(t) + b_0x(t)$$

in the sinusoidal steady state. Assume $a_2 = 1$, $a_1 = 0.5$, $a_0 = 1$, $b_2 = 4$, $b_1 = 0$, $b_0 = 4$.

3.8 Consider the system from Problem 3.6.

(a) Find the frequency response.

(b) Plot the magnitude response in decibels, the phase response, and the group delay.

(c) Compute the poles and zeros of the system. For each complex pole find the pole magnitude and the pole quality factor.

3.9 The frequency response of a system is given by

$$H(j\omega) = \frac{9 - \omega^2}{2(4 + 5j\omega - 2\omega^2 - j\omega^3)}$$

(a) Plot the magnitude response in linear and log scale.

(b) Resolve the system into first-order sections and biquads.

(c) Find the frequency at which the magnitude response is 3 dB below the maximum.

3.10 Consider the system in Fig. 3.1 and assume identical integrators, characterized by $y_{int}(t) = \int x_{int}(t)\,dt$, where $x_{int}(t)$ is integrator input and $y_{int}(t)$ is integrator output. The system input is at node 1 and the output is at node 4.

(a) Find the frequency response of the system.

(b) Compute phasors and steady-state signals at all nodes if the system is excited by $x(t) = \sin(\pi t)$. Assume $Q = 1$.

(c) Can this system reach a steady state for an arbitrary sinusoidal input?

3.11 Circuit in Fig. 3.4 is in steady state.

(a) Find the frequency response of the circuit (assume that the output is at node 4).

(b) Derive the input impedance seen by the generator.

(c) Compute phasors and steady-state voltages at all nodes if the circuit is excited by $v_g(t) = \cos(5000\pi t)$. Assume $R_1 = R_3 = 10\,\mathrm{k\Omega}$, $C_2 = 40\,\mathrm{nF}$, $C_4 = 10\,\mathrm{nF}$.

(d) Can this circuit reach a steady state for an arbitrary sinusoidal input?

(e) Use phasor method to find the steady-state output for $v_g(t) = 1 + \sin(5000\pi t)$.

3.12 Derive the Fourier series for a sawtooth signal which rises linearly from 0 to A over the range $0 \le t \le T$, then drops instantaneously to 0 and repeats the cycle. Find the amplitude spectrum of the sawtooth signal. Assume $A = 5$, $T = 2$.

3.13 Use the results of Problem 3.12 to deduce the amplitude spectrum of the signal $x(t) = At$ for $|t| < \frac{1}{2}T$, $x(t + kT) = x(t)$ for integer k.

3.14 Derive the Fourier series for a unit periodic square pulse train defined as

$$x(t) = \begin{cases} 1, & 0 \le t < \dfrac{T}{2} \\ -1, & \dfrac{T}{2} \le t < T \end{cases}$$

$x(t + kT) = x(t)$ for integer k. Assume $T = 2$.

Approximate $x(t)$ with a finite sum of the first five harmonics. Plot the approximation and observe an overshoot at points very near the corners $t = k\dfrac{T}{2}$.

3.15 Consider a train of square pulses

$$x(t) = \begin{cases} A, & 0 < t \le \tau \\ 0, & \tau < t \le T \end{cases}$$

$x(t + kT) = x(t)$ for integer k. Derive the Fourier series in the complex form (the exponential Fourier series). Assume $A = 1/\tau$ and find the complex Fourier coefficients for very short pulses when τ tends to zero.

3.16 A periodic signal is given by

$$x(t) = A\frac{t^2}{T^2}, \qquad 0 \le t < T$$

where $x(t + kT) = x(t)$ for integer k. Derive the trigonometric Fourier series and the exponential Fourier series. Observe the relation between the Fourier coefficients. Assume $A = 1$, $T = 1$.

3.17 Employ the definition of the Fourier transform to compute the transform of the signal

$$x(t) = A\cos(\pi\frac{t}{T})^2, \qquad |t| < \frac{T}{2}$$

$$x(t) = 0, \qquad |t| \ge \frac{T}{2}$$

Assume $A = 1$, $T = 1$.

3.18 Find the Laplace transform of the signals

(a) $x(t) = A\cos(\pi\dfrac{t}{T})^2\, u(t)$

(b) $x(t) = A\exp(-\alpha t)\,\cos(\omega t + \xi)u(t)$.

Assume $A = 2$, $T = 1$, $\alpha = 0.1$, $\omega = 100\pi$, $\xi = 0.2\pi$.

3.19 The system from Problem 3.5 is excited by $x(t) = tu(t)$. Find the zero-state response. Use the Laplace transform.

3.20 Find the zero-state response of the system from Problem 3.6 for the causal input $x(t) = (X_m\cos(\omega t) + X_m\cos(2\omega t))\,u(t)$. Assume $X_m = \sqrt{2}$, $\omega = 2$.

3.21 Derive the transfer function in the Laplace domain for the system specified in Problem 3.7. What is the value of the impulse response of the system at $t \to +\infty$?

3.22 The Laplace transform of a signal is given by

$$X(s) = A\frac{s + a}{s(s + b)}, \qquad a \ne b$$

Does this signal vanish as $t \to +\infty$? Assume $A = 4$, $a = 2$, $b = 3$.

3.23 Use the Laplace transform to find the output of the system specified in Problem 3.10 for the input $x(t) = \cos(\pi t)\, u(t)$, and the initial conditions $y_3(0^-) = K_3 = 1.25$ and $y_4(0^-) = K_4 = -0.75$. Try to solve this problem in the time domain. Compute the impulse response of the system.

3.24 Find the step response of the circuit specified in Problem 3.11. Use the Laplace transform. Compute the zero-input response if the voltage across C_2 has a nonzero value $(v_2 - v_4)_{t=0-} = K = 0.025$.

3.25 Consider the periodic signal

$$x(t) = 4\sin(2\pi t) + \sin(200\pi t)$$

Sample this signal uniformly at n points over the interval $0 \le t < 1$, and find the DFT for (a) $n = 8$, (b) $n = 32$, (c) $n = 64$, (d) $n = 128$, (e) $n = 256$. Plot and compare the signal spectrum for each n.

3.26 Find the transfer function of the digital filter shown in Fig. 3.2 and its poles and zeros.

3.27 Compute the zero-input response of the digital filter in Fig. 3.2, if the initial conditions are $y_5(-1) = 0.2$, $y_{10}(-1) = 0.1$.

■ MATLAB EXERCISES

3.1 Write a MATLAB script that finds and plots sinusoids corresponding to the phasors in Problem 3.3.

3.2 Write a MATLAB script that performs the computation required in Problem 3.4. Plot $x(t)$ and $x_e(t)$ on the same diagram, for $0 \le t \le 2$, to validate the equivalence.

3.3 Find and plot the steady-state response from Problem 3.5 in MATLAB.

3.4 Write a MATLAB script that finds and plots the steady-state response from Problem 3.6.

3.5 Plot the magnitude response and the phase response from Problem 3.7 for $a_2 = 1$, $a_1 = 2$, $a_0 = 5$, $b_2 = 3$, $b_1 = 4$, $b_0 = 6$, by using the MATLAB function `freqs`.

3.6 Write a MATLAB script that carries out the computation required in Problem 3.8.

3.7 Can you automate the analysis from Problem 3.9 in MATLAB? If yes, write the corresponding code.

3.8 Write a MATLAB script for computation of the Fourier coefficients for signals from Problems 3.12, 3.13, 3.14, 3.15, 3.16.

(a) Write your own code from the scratch (without the `fft` function).

(b) Use the MATLAB function `fft`.

3.9 Find the Fourier transform from Problem 3.17 in MATLAB. Use the MATLAB function `fft`.

■ *MATHEMATICA* EXERCISES

3.1 Write the corresponding code in *Mathematica* for symbolic computation of the phasor transform in Problem 3.1.

3.2 Find and plot sinusoids from Problem 3.3 in *Mathematica*.

3.3 Write a *Mathematica* notebook that performs the computation required in Problem 3.4. Plot $x(t)$ and $x_e(t)$ on the same diagram, for $0 \le t \le 2$, to validate the equivalence.

3.4 Write a *Mathematica* code that finds the steady-state response from Problem 3.5.

3.5 Find symbolically the steady-state response from Problem 3.6 in *Mathematica*.

3.6 Automate the manipulation from Problem 3.7 in *Mathematica*, and find the transfer function symbolically. Plot the magnitude response and the phase response for $a_2 = 1$, $a_1 = 2$, $a_0 = 5$, $b_2 = 3$, $b_1 = 4$, $b_0 = 6$, $0 \leq \omega \leq 8$.

3.7 Analyze the system from Problem 3.8 symbolically in *Mathematica*. Find the required quantities symbolically.

3.8 Use *Mathematica* for symbolic analysis of the system from Problem 3.9. Try to find the required quantities symbolically.

3.9 Write a *Mathematica* notebook to automate the analysis required in Problem 3.10.

3.10 Consider the circuit from Problem 3.11.

- **(a)** Carry out the general symbolic analysis in *Mathematica*. Formulate the circuit equations in terms of phasors, find the frequency response, and compute the phasors for all node voltages.
- **(b)** Plot the magnitude response, phase response, and group delay.
- **(c)** Find the steady-state node voltages.

3.11 Write a *Mathematica* code to derive the Fourier series. Next, find the Fourier series for signals from Problems 3.12, 3.13, 3.14, 3.15, and 3.16.

3.12 Use *Mathematica* to derive the Fourier transform in Problem 3.17.

- **(a)** Write your own code.
- **(b)** Use the *Mathematica* standard packages.

3.13 Automate the Laplace domain manipulation from Problems 3.18, 3.19, 3.20, 3.21, 3.22, 3.23, and 3.24 in *Mathematica*.

- **(a)** Write your own code.
- **(b)** Use the *Mathematica* standard packages.