CHAPTER 6

ADVANCED ANALOG FILTER DESIGN ALGORITHMS

Classical analog filter design techniques return only one design from an infinite collection of alternative designs, or fail to design filters when solutions exist. These classic techniques hide a wealth of alternative filter designs that are more robust when implemented in analog circuits. In this chapter, we present (1) case studies of optimal analog filters that cannot be designed with classic techniques and (2) the formal, mathematical framework that underlies their solutions. We have automated the advanced filter design techniques in software, so we present detailed step-by-step design algorithms.

6.1 INTRODUCTION

In designing analog filters, one generally relies on canned software routines or mechanical table-oriented procedures. The primary reason for these "black box" approaches is that the approximation theory that underlies filter design includes complex mathematics. Unfortunately, conventional approaches return only one design, thereby hiding a wealth of alternative filter designs that are more robust when implemented in analog circuits. In addition, conventional approaches may fail to find a filter when in fact one exists.

We develop advanced design techniques to find a comprehensive set of optimal designs to represent the infinite solution space. The optimal designs include filters that have minimal order, minimal quality factors, minimal complexity, minimal sensitivity to pole-zero locations, minimal deviation from a specified group delay, approximate linear phase response, and minimized peak overshoot of the step response. We base our approach on formal, mathematical properties of Jacobi elliptic functions [47, 53]. We automate these advanced filter design techniques in software [2, 51].

The key observations underlying advanced analog filter design are that

1. many designs satisfy the same user specification;
2. Butterworth and Chebyshev filters are special cases of elliptic filters; and
3. minimum-order filters may not be as efficient to implement as some higher-order filters.

The filter optimization problem is a mixed-integer linear programming problem so the classical techniques break down. Instead of using iterative numerical techniques, we solve these problems using closed-form algebraic expressions. Then, we present several new case studies of optimal analog filters that cannot be designed with classical techniques, and the formal, mathematical framework that underlies their solutions.

6.2 NOTATION

We review the list of symbols that we use in formulas and procedures when designing analog filters. Often, we append a suffix to designate a quantity related to a specific filter type. For example, we add h to designate highpass filter; thus, A_{ph} is A_p of a highpass filter.

$A(\omega)$—attenuation (dB)

A_p—maximum passband attenuation in specification (dB)

A_s—minimum stopband attenuation in specification (dB)

$\mathrm{cd}(u, k)$—Jacobi elliptic cd function

$\mathrm{cd}^{-1}(v, k)$—inverse Jacobi elliptic cd function

f—frequency (Hz)

f_{nQ}—normalized frequency in minimal Q-factor design

f_p—passband edge frequency of designed filter (Hz)

F_p—passband edge frequency in specification (Hz)

f_s—stopband edge frequency of designed filter (Hz)

F_s—stopband edge frequency in specification (Hz)

$G(\omega)$—gain (dB)

$h(H(s), t)$—impulse response

$h_s(H(s), t)$—step response

$\mathcal{H}(n, \xi, \epsilon, p)$—normalized lowpass transfer function

$\mathcal{H}_{\min Q}(n, \xi, p)$—minimal Q-factor normalized lowpass transfer function

$H(s)$—transfer function

i—index ($i = 1, 2, \ldots, n$)

j—the imaginary unit ($j = \sqrt{-1}$)

k—modulus of elliptic functions

$K_e(n, \xi, \epsilon, x)$—elliptic characteristic function

$K_I(k)$—complete elliptic integral of first kind
K_p—characteristic function passband specification
K_s—characteristic function stopband specification
$L(n, \xi)$—discrimination factor
$\mathcal{L}^{-1}(H(s))$—the inverse Laplace transform of $H(s)$
$M(\omega)$—magnitude response, $M(\omega) = |H(j\omega)|$
n—transfer function order (order for short)
$n_{but}(F_p, F_s, K_p, K_s)$—minimum Butterworth order
$n_{cheb}(F_p, F_s, K_p, K_s)$—minimum Chebyshev order
$n_{ellip}(F_p, F_s, K_p, K_s)$—minimum elliptic order
$n_{\max}$—maximum order
$n_{\min}$—minimum order
$n_{minQ}(F_p, F_s, K_p, K_s)$—minimum order of minimal Q-factor design
p—normalized complex frequency
$q(k)$—modular constant
$Q(s_i)$—quality factor of pole/zero s_i
$Q_{\min Q}(n, \xi, i)$—quality factor of ith pole of minimal Q-factor design
$R(n, \xi, x)$—elliptic rational function
s—complex frequency, Laplace operator (rad/s)
$\mathrm{sn}(u, k)$—Jacobi elliptic sn function
$\mathrm{sn}^{-1}(v, k)$—inverse Jacobi elliptic sn function
$S(n, \xi, \epsilon, i)$—ith pole of normalized lowpass transfer function
S_A—attenuation-limit specification
S_G—gain-limit specification
S_K—characteristic-function-limit specification
S_M—magnitude-limit specification
S_r—magnitude-ripple specification
S_δ—magnitude-tolerance specification
$S_{minQ}(n, \xi, i)$—ith pole of normalized lowpass transfer function for minimal Q-factor design
t—time (s)
x—dimensionless variable
$X(n, \xi, i)$—ith zero of elliptic rational function
δ_1—passband magnitude ripple
δ_2—stopband magnitude ripple
δ_p—passband magnitude tolerance
δ_s—stopband magnitude tolerance
$\zeta(n, \xi, \epsilon)$—auxiliary function
ϵ—ripple factor
ξ—selectivity factor

$\tau_{GD}(H(s), \omega)$—group delay (s)

$\Phi(\omega)$—phase response, $\Phi(\omega) = \arg(H(j\omega))$

ω—angular frequency (rad/s), $\omega = 2\pi f$

$\lfloor x \rfloor$—integer, $x \le \lfloor x \rfloor < x + 1$

$\underset{x_1<x<x_2}{\text{FindRoot}} \{ F(x) = G(x) \}$ find real x over interval $x_1 < x < x_2$ by solving $F(x) = G(x)$

$\underset{\substack{x_1<x<x_2 \\ y_1<y<y_2}}{\text{FindRoot}} \begin{Bmatrix} F_1(x) = G_1(x) \\ F_2(x) = G_2(x) \end{Bmatrix}$ find real x over interval $x_1 < x < x_2$ and real y over interval $y_1 < y < y_2$ by solving set of equations $\{F_1(x) = G_1(x),\ F_2(x) = G_2(x)\}$

6.3 DESIGN EQUATIONS AND PROCEDURES

In this section we summarize all design equations, formulas, and procedures that are based on Jacobi elliptic functions. We use this relations in the purely numerical design.

6.3.1 Specification

A design specification can be given in different ways:

$$\boxed{S_A = \{F_p, F_s, A_p, A_s\}} \tag{6.1}$$

$$\boxed{S_G = \{F_p, F_s, G_p, G_s\}} \tag{6.2}$$

$$\boxed{S_\delta = \{F_p, F_s, \delta_p, \delta_s\}} \tag{6.3}$$

$$\boxed{S_M = \{F_p, F_s, M_p, M_s\}} \tag{6.4}$$

$$\boxed{S_r = \{F_p, F_s, \delta_1, \delta_2\}} \tag{6.5}$$

$$\boxed{S_K = \{F_p, F_s, K_p, K_s\}} \tag{6.6}$$

We use a set of functions to convert one form of specification into another.

$$\boxed{\begin{aligned} K_p(A_p) &= \frac{\sqrt{1 - 10^{-A_p/10}}}{10^{-A_p/20}} \\ K_s(A_s) &= \frac{\sqrt{1 - 10^{-A_s/10}}}{10^{-A_s/20}} \end{aligned}} \tag{6.7}$$

$$\boxed{\begin{aligned} K_p(G_p) &= \frac{\sqrt{1 - 10^{G_p/10}}}{10^{G_p/20}} \\ K_s(G_s) &= \frac{\sqrt{1 - 10^{G_s/10}}}{10^{G_s/20}} \end{aligned}} \tag{6.8}$$

$$\boxed{\begin{aligned} K_p(\delta_p) &= \frac{\sqrt{\delta_p(2 - \delta_p)}}{1 - \delta_p} \\ K_s(\delta_s) &= \frac{\sqrt{1 - \delta_s^2}}{\delta_s} \end{aligned}} \tag{6.9}$$

$$\boxed{\begin{aligned} K_p(M_p) &= \frac{\sqrt{1 - M_p^2}}{M_p} \\ K_s(M_s) &= \frac{\sqrt{1 - M_s^2}}{M_s} \end{aligned}} \tag{6.10}$$

$$\boxed{\begin{aligned} K_p(\delta_1) &= \frac{2\sqrt{\delta_1}}{1 - \delta_1} \\ K_s(\delta_1, \delta_2) &= \frac{\sqrt{(1 + \delta_1)^2 - \delta_2^2}}{\delta_2} \end{aligned}} \tag{6.11}$$

6.3.2 Special and Auxiliary Functions

The following special mathematical functions are used in the design of elliptic filters:

$$K_J(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \tag{6.12}$$

$$\boxed{\begin{aligned} v &= \mathrm{sn}^{-1}\left(\frac{1}{\sqrt{1 + \epsilon^2}}, \sqrt{1 - \frac{1}{L^2(n, \xi)}}\right) \\ \zeta(n, \xi, \epsilon) &= \mathrm{sn}\left(\frac{K_J\left(\sqrt{1 - \frac{1}{\xi^2}}\right)}{K_J\left(\sqrt{1 - \frac{1}{L^2(n, \xi)}}\right)} v, \sqrt{1 - \frac{1}{\xi^2}}\right) \end{aligned}} \tag{6.13}$$

The procedure given by Eq. (6.13) comes from Eq. (12.302).

$$
\boxed{
\begin{array}{ll}
\begin{aligned}
t &= \frac{1}{2}\frac{1-\sqrt[4]{1-k^2}}{1+\sqrt[4]{1-k^2}} \\
q' &= t + 2t^5 + 15t^9 + 150t^{13} + 1\,707t^{17} \\
&\quad + 20{,}910t^{21} + 268{,}616t^{25} + 3{,}567{,}400t^{29}, \\
&\quad + 48{,}555{,}069t^{33} + 673{,}458{,}874t^{37} \\
q(k) &= q'
\end{aligned} & k \le \dfrac{1}{\sqrt{2}} \\[3ex]
\begin{aligned}
t &= \frac{1}{2}\frac{1-\sqrt{k}}{1+\sqrt{k}} \\
q' &= t + 2t^5 + 15t^9 + 150t^{13} + 1{,}707t^{17} \\
&\quad + 20{,}910t^{21} + 268{,}616t^{25} + 3{,}567{,}400t^{29}, \\
&\quad + 48{,}555{,}069t^{33} + 673{,}458{,}874t^{37} \\
q(k) &= e^{\pi^2/\ln q'}
\end{aligned} & \dfrac{1}{\sqrt{2}} < k < 1
\end{array}
}
\tag{6.14}
$$

The procedure given by Eq. (6.14) comes form Eqs. (12.60), (12.64)–(12.67).

6.3.3 Transfer Function Order

For a given specification we compute the minimal order of the approximation functions.

$$
\boxed{
\begin{aligned}
k &= \frac{F_p}{F_s} \\
L &= \frac{K_s}{K_p} \\
N &= \frac{K_J\left(\sqrt{1-\frac{1}{L^2}}\right)}{K_J\left(\frac{1}{L}\right)} \\
D &= \frac{K_J\left(\sqrt{1-k^2}\right)}{K_J(k)} \\
n_{ellip}(F_p, F_s, K_p, K_s) &= \left\lceil \frac{N}{D} \right\rceil
\end{aligned}
}
\tag{6.15}
$$

The procedure given by Eq (6.15) comes from Eq. (12.338).

$$\boxed{\begin{aligned} \xi &= \frac{F_p}{F_s} \\ L &= \frac{K_s}{K_p} \\ N &= \cosh^{-1} L \\ D &= \cosh^{-1} \frac{1}{\xi} \\ n_{cheb}\left(F_p, F_s, K_p, K_s\right) &= \left\lfloor \frac{N}{D} \right\rfloor \end{aligned}} \tag{6.16}$$

$$\boxed{\begin{aligned} \xi &= \frac{F_p}{F_s} \\ L &= \frac{K_s}{K_p} \\ N &= \log_{10} L \\ D &= \log_{10} \frac{1}{\xi} \\ n_{but}\left(F_p, F_s, K_p, K_s\right) &= \left\lfloor \frac{N}{D} \right\rfloor \end{aligned}} \tag{6.17}$$

The procedures given by Eqs. (6.16) and (6.17) are based on formulas given in reference 15.

$$\boxed{\begin{aligned} n_{\min} &= n_{ellip}\left(F_p, F_s, K_p, K_s\right) \\ n_{\max} &= 2\, n_{ellip}\left(F_p, F_s, K_p, K_s\right) \end{aligned}} \tag{6.18}$$

$$\boxed{\begin{aligned} i &= n_{ellip}\left(F_p, F_s, K_p, K_s\right) \\ \xi_i &= \xi_{min}(i, 1, K_s^2) \\ &\boxed{\begin{aligned} &\text{While } K_e\left(i, \xi_i, \frac{1}{K_s}, \frac{F_s}{F_p}\xi_i\right) > K_p \\ &i = i + 1 \\ &\xi_i = \xi_{min}(i, 1, K_s^2) \end{aligned}} \\ n_{\min Q}\left(F_p, F_s, K_p, K_s\right) &= i \end{aligned}} \tag{6.19}$$

The procedure given by Eq. (6.19) comes from Eqs. (12.338) and (12.367)–(12.369).

6.3.4 Zeros, Poles, and Q-Factors

The basic functions for the elliptic design are $X(n,\xi,i)$, $S(n,\xi,\epsilon,i)$, and $Q(s)$, as follows:

$$\boxed{\begin{aligned} X(n,\xi,i) &= -\mathrm{cd}\left(\frac{2i-1}{n}K_J(\frac{1}{\xi}),\frac{1}{\xi}\right)\\ X(n,\xi,(n+1)/2) &= 0, \quad n \text{ odd}\\ X(n,\xi,1) < X(n,\xi,2) < &\ \ldots < X(n,\xi,n) \end{aligned}} \tag{6.20}$$

The procedure given by Eq. (6.20) comes from Eqs. (12.134) and (12.323).

$$\boxed{\begin{aligned} \zeta &= \zeta(n.\xi,\epsilon)\\ x &= X(n,\xi,i)\\ N_{re} &= -\zeta\sqrt{1-\zeta^2}\sqrt{1-x^2}\sqrt{1-\frac{x^2}{\xi^2}}\\ N_{im} &= x\sqrt{1-\left(1-\frac{1}{\xi^2}\right)\zeta^2}\\ N &= N_{re}+jN_{im}\\ D &= 1-\left(1-\frac{x^2}{\xi^2}\right)\zeta^2\\ S(n,\xi,\epsilon,i) &= \frac{N}{D} \end{aligned}} \tag{6.21}$$

The procedure given by Eq. (6.21) comes from Eq. (12.266).

$$\boxed{\begin{aligned} N_{re} &= -\sqrt{1-X(n,\xi,i)^2}\sqrt{\xi^2-X(n,\xi,i)^2}\\ N_{im} &= X(n,\xi,i)(\xi+1)\\ N &= N_{re}+jN_{im}\\ D &= \xi+X(n,\xi,i)^2\\ S_{\min Q}(n,\xi,i) &= \sqrt{\xi}\frac{N}{D} \end{aligned}} \tag{6.22}$$

The procedure given by Eq. (6.22) comes from Eq. (12.370).

$$\boxed{Q(s) = -\frac{|s|}{2\mathrm{Re}(s)}} \tag{6.23}$$

$$\boxed{Q_{\min Q}(n,\xi,i) = \frac{\xi+X(n,\xi,i)^2}{2\sqrt{1-X(n,\xi,i)^2}\sqrt{\xi^2-X(n,\xi,i)^2}}} \tag{6.24}$$

Equation (6.24) comes from Eq. (12.372).

6.3.5 Discrimination Factor, Elliptic Rational Function, and Characteristic Function

The essential functions for the elliptic approximation are $L(n,\xi)$, $R(n,\xi,x)$, and $K_c(n,\xi,\epsilon,x)$:

$$L(n,\xi) = \frac{1}{\xi^n}\frac{\prod_{i=1}^{n/2}\left(\xi^2 - X^2(n,\xi,i)\right)^2}{\prod_{i=1}^{n/2}\left(1 - X^2(n,\xi,i)\right)^2} \qquad \begin{matrix} n \text{ even} \\ n=2,4,\ldots \end{matrix}$$

$$L(n,\xi) = \frac{1}{\xi^{n-2}}\frac{\prod_{i=1}^{(n-1)/2}\left(\xi^2 - X^2(n,\xi,i)\right)^2}{\prod_{i=1}^{(n-1)/2}\left(1 - X^2(n,\xi,i)\right)^2} \qquad \begin{matrix} n \text{ odd} \\ n=1,3,\ldots \end{matrix} \tag{6.25}$$

The procedure given by Eq. (6.25) comes from Eqs. (12.359) and (12.360).

$$r_0 = \frac{\prod_{i=1}^{n/2}\left(1 - X^2(n,\xi,i)\right)}{\prod_{i=1}^{n/2}\left(1 - \dfrac{\xi^2}{X^2(n,\xi,i)}\right)} \qquad \begin{matrix} n \text{ even} \\ n=2,4,\ldots \end{matrix}$$

$$R(n,\xi,x) = \frac{1}{r_0}\frac{\prod_{i=1}^{n/2}\left(x^2 - X^2(n,\xi,i)\right)}{\prod_{i=1}^{n/2}\left(x^2 - \dfrac{\xi^2}{X^2(n,\xi,i)}\right)}$$

$$r_0 = \frac{\prod_{i=1}^{(n-1)/2}\left(1 - X^2(n,\xi,i)\right)}{\prod_{i=1}^{(n-1)/2}\left(1 - \dfrac{\xi^2}{X^2(n,\xi,i)}\right)} \qquad \begin{matrix} n \text{ odd} \\ n=1,3,\ldots \end{matrix} \tag{6.26}$$

$$R(n,\xi,x) = \frac{1}{r_0}\frac{x\prod_{i=1}^{(n-1)/2}\left(x^2 - X^2(n,\xi,i)\right)}{\prod_{i=1}^{(n-1)/2}\left(x^2 - \dfrac{\xi^2}{X^2(n,\xi,i)}\right)}$$

The procedure given by Eq. (6.26) comes from Eqs. (12.141)–(12.144).

$$\boxed{K_e(n,\xi,\epsilon,x) = \epsilon\,|R(n,\xi,x)|} \tag{6.27}$$

6.3.6 Normalized Lowpass Elliptic Transfer Function

Our goal is to find the transfer function that meets a lowpass specification.

$$g = \frac{1}{\sqrt{1+\epsilon^2}} \frac{\prod_{i=1}^{n/2} |S(n,\xi,\epsilon,i)|^2}{\prod_{i=1}^{n/2} \frac{\xi^2}{X^2(n,\xi,i)}} \qquad \begin{matrix} n \text{ even} \\ n=2,4,\ldots \end{matrix}$$

$$\mathcal{H}(n,\xi,\epsilon,p) = \frac{g \prod_{i=1}^{n/2} p^2 + \frac{\xi^2}{X^2(n,\xi,i)}}{\prod_{i=1}^{n/2} p^2 - 2p\mathrm{Re}\,(S(n,\xi,\epsilon,i)) + |S(n,\xi,\epsilon,i)|^2}$$

$$g = -S\left(n,\xi,\epsilon,\frac{n+1}{2}\right) \frac{\prod_{i=1}^{\frac{n-1}{2}} |S(n,\xi,\epsilon,i)|^2}{\prod_{i=1}^{\frac{n-1}{2}} \frac{\xi^2}{X^2(n,\xi,i)}} \qquad \begin{matrix} n \text{ odd} \\ n=1,3,\ldots \end{matrix}$$

$$\mathcal{H}(n,\xi,\epsilon,p) = \frac{\frac{g}{p - S\left(n,\xi,\epsilon,\frac{n+1}{2}\right)} \prod_{i=1}^{\frac{n-1}{2}} p^2 + \frac{\xi^2}{X^2(n,\xi,i)}}{\prod_{i=1}^{\frac{n-1}{2}} p^2 - 2p\mathrm{Re}\,S(n,\xi,\epsilon,i) + |S(n,\xi,\epsilon,i)|^2} \tag{6.28}$$

The procedure given by Eq. (6.28) comes from Eq. (13.196).

$$g = \frac{\prod_{i=1}^{n/2} X(n,\xi,i)^2}{\sqrt{\xi^n}\sqrt{1+\frac{1}{L(n,a)}}} \qquad \begin{array}{l} n \text{ even} \\ n=2,4,\ldots \end{array}$$

$$\mathcal{H}_{\min Q}(n,\xi,\epsilon,p) = g\frac{\prod_{i=1}^{n/2} p^2 + \frac{\xi^2}{X^2(n,\xi,i)}}{\prod_{i=1}^{n/2} p^2 + p\frac{\sqrt{\xi}}{Q_{\min Q}(n,\xi,i)} + \xi}$$

$$g = \frac{\prod_{i=1}^{\frac{n-1}{2}} X(n,\xi,i)^2}{\sqrt{\xi^{n-2}}\sqrt{1+\frac{1}{L(n,a)}}} \qquad \begin{array}{l} n \text{ odd} \\ n=1,3,\ldots \end{array}$$

$$\mathcal{H}_{\min Q}(n,\xi,\epsilon,p) = g\frac{1}{p+\sqrt{\xi}}\frac{\prod_{i=1}^{\frac{n-1}{2}} p^2 + \frac{\xi^2}{X^2(n,\xi,i)}}{\prod_{i=1}^{\frac{n-1}{2}} p^2 + p\frac{\sqrt{\xi}}{Q_{\min Q}(n,\xi,i)} + \xi} \tag{6.29}$$

The procedure given by Eq. (6.29) comes from Eqs. (12.373) and (12.374).

6.3.7 Selectivity Factor, Ripple Factor, and Edge Frequencies

We compute the boundary values of the design space from

$$\boxed{\begin{aligned} L &= \frac{K_s}{K_p} \\ g &= \left(q\left(\frac{1}{L}\right)\right)^{1/n} \\ N &= 1 + 2\sum_{m=1}^{9}(-1)^m g^{m^2} \\ D &= 1 - 2\sum_{m=1}^{9} g^{m^2} \\ \xi_{min}\left(n, K_p, K_s\right) &= \frac{1}{\sqrt{1-\left(\frac{N}{D}\right)^4}} \end{aligned}} \tag{6.30}$$

The procedure given by Eq. (6.30) comes from Eq. (12.354).

$$\boxed{\xi_{\max}\left(n, F_p, F_s, K_p, K_s\right) = x \Big| \underset{F_s/F_p < x < 10F_s/F_p}{\text{FindRoot}} \; R(n, x, \frac{F_s}{F_p}) = \frac{K_s}{K_p}} \tag{6.31}$$

$$\boxed{1 < \xi_{\min} < \frac{F_s}{F_p} < \xi_{\max} < \infty} \tag{6.32}$$

$$\boxed{\epsilon_{\min}\left(n, F_p, F_s, K_p, K_s\right) = \frac{K_s}{L\left(n, \xi_{\max}\left(n, F_p, F_s, K_p, K_s\right)\right)}} \tag{6.33}$$

$$\boxed{\epsilon_{\max}\left(K_p\right) = K_p} \tag{6.34}$$

$$\boxed{f_{p,min}\left(n, F_p, F_s, K_p, K_s\right) = \frac{F_s}{\xi_{\max}\left(n, F_p, F_s, K_p, K_s\right)}} \tag{6.35}$$

$$\boxed{f_{p,\max}\left(n, F_s, K_p, K_s\right) = \frac{F_s}{\xi_{min}\left(n, K_p, K_s\right)}} \tag{6.36}$$

$$\boxed{\begin{aligned}
\xi_h &= \xi_{\max}(n, F_p, F_s, K_p, K_s) \\
\xi_l &= x \Bigg|_{\substack{\text{FindRoot} \\ \sqrt{\xi_h \frac{F_s}{F_p}} < x < \xi_h}} K_e\left(n, x, \frac{1}{\sqrt{L(n,x)}}, x\frac{F_s}{F_p}\right) = K_p \\
f_l &= x \Bigg|_{\substack{\text{FindRoot} \\ \sqrt[4]{\xi_l} < x < \sqrt{\xi_l}}} K_e\left(n, \xi_l, \frac{1}{\sqrt{L(n,\xi_l)}}, x\right) = K_p \\
f_h &= x \Bigg|_{\substack{\text{FindRoot} \\ \sqrt[4]{\xi_h} < x < \sqrt{\xi_h}}} K_e\left(n, \xi_h, \frac{1}{\sqrt{L(n,\xi_h)}}, x\right) = K_p \\
\begin{array}{r} \xi_{\min Q}(n, F_p, F_s, K_p, K_s) = x \\ f_{nQ}(n, F_p, F_s, K_p, K_s) = y \end{array} &\Bigg|_{\substack{\text{FindRoot} \\ \xi_l < x < \xi_h \\ f_l < y < f_h}} \left\{ \begin{array}{l} K_e\left(n, x, \frac{1}{\sqrt{L(n,x)}}, y\right) = K_p \\ K_e\left(n, x, \frac{1}{\sqrt{L(n,x)}}, y\frac{F_s}{F_p}\right) = K_s \end{array} \right\}
\end{aligned}} \tag{6.37}$$

6.4 DESIGN D1

1. Start from a specification and convert it into the characteristic-function-limit specification

$$\left.\begin{aligned}
S_A &= \{F_p, F_s, A_p, A_s\} \\
S_\delta &= \{F_p, F_s, \delta_p, \delta_s\} \\
S_M &= \{F_p, F_s, M_p, M_s\} \\
S_r &= \{F_p, F_s, \delta_1, \delta_2\} \\
S_G &= \{F_p, F_s, G_p, G_s\}
\end{aligned}\right\} \longmapsto S_K = \{F_p, F_s, K_p, K_s\} \tag{6.38}$$

2. Compute the minimal order $n_{\min} = n_{ellip}(F_p, F_s, K_p, K_s)$.
3. Choose the order

$$n \geq n_{\min} \tag{6.39}$$

4. Compute the selectivity factor

$$\xi = \frac{F_s}{F_p} \tag{6.40}$$

5. Choose the ripple factor

$$\epsilon = K_p \tag{6.41}$$

6. Choose the actual passband edge

$$f_p = F_p \tag{6.42}$$

7. Construct the normalized lowpass transfer function

$$\mathcal{H}(n, \xi, \epsilon, p) \tag{6.43}$$

8. Construct the lowpass transfer function

$$H(s) = \mathcal{H}\left(n, \xi, \epsilon, \frac{s}{2\pi f_p}\right) \tag{6.44}$$

6.5 DESIGN D2

1. Start from a specification and convert it into the characteristic-function-limit specification

$$\left.\begin{aligned} S_A &= \{F_p, F_s, A_p, A_s\} \\ S_\delta &= \{F_p, F_s, \delta_p, \delta_s\} \\ S_M &= \{F_p, F_s, M_p, M_s\} \\ S_r &= \{F_p, F_s, \delta_1, \delta_2\} \\ S_G &= \{F_p, F_s, G_p, G_s\} \end{aligned}\right\} \longmapsto S_K = \{F_p, F_s, K_p, K_s\} \tag{6.45}$$

2. Compute the minimal order $n_{\min} = n_{ellip}(F_p, F_s, K_p, K_s)$.
3. Choose the order

$$n \geq n_{\min} \tag{6.46}$$

4. Compute the selectivity factor

$$\xi = \frac{F_s}{F_p} \tag{6.47}$$

5. Compute the ripple factor

$$\epsilon = \frac{K_s}{L(n, \xi)} \tag{6.48}$$

6. Choose the actual passband edge

$$f_p = F_p \tag{6.49}$$

7. Construct the normalized lowpass transfer function

$$\mathcal{H}(n, \xi, \epsilon, p) \tag{6.50}$$

8. Construct the lowpass transfer function

$$H(s) = \mathcal{H}\left(n, \xi, \epsilon, \frac{s}{2\pi f_p}\right) \tag{6.51}$$

6.6 DESIGN D3A

1. Start from a specification and convert it into the characteristic-function-limit specification

$$\left.\begin{aligned} S_A &= \{F_p, F_s, A_p, A_s\} \\ S_\delta &= \{F_p, F_s, \delta_p, \delta_s\} \\ S_M &= \{F_p, F_s, M_p, M_s\} \\ S_r &= \{F_p, F_s, \delta_1, \delta_2\} \\ S_G &= \{F_p, F_s, G_p, G_s\} \end{aligned}\right\} \longmapsto S_K = \{F_p, F_s, K_p, K_s\} \tag{6.52}$$

2. Compute the minimal order $n_{\min} = n_{ellip}(F_p, F_s, K_p, K_s)$.
3. Choose the order

$$n \geq n_{\min} \tag{6.53}$$

4. Compute the selectivity factor

$$\xi = \xi_{min}(n, K_p, K_s) \tag{6.54}$$

5. Choose the ripple factor

$$\epsilon = K_p \tag{6.55}$$

6. Choose the actual passband edge

$$f_p = F_p \tag{6.56}$$

7. Construct the normalized lowpass transfer function

$$\mathcal{H}(n, \xi, \epsilon, p) \tag{6.57}$$

8. Construct the lowpass transfer function

$$H(s) = \mathcal{H}\left(n, \xi, \epsilon, \frac{s}{2\pi f_p}\right) \tag{6.58}$$

6.7 DESIGN D3B

1. Start from a specification and convert it into the characteristic-function-limit specification

$$\left.\begin{aligned} S_A &= \{F_p, F_s, A_p, A_s\} \\ S_\delta &= \{F_p, F_s, \delta_p, \delta_s\} \\ S_M &= \{F_p, F_s, M_p, M_s\} \\ S_r &= \{F_p, F_s, \delta_1, \delta_2\} \\ S_G &= \{F_p, F_s, G_p, G_s\} \end{aligned}\right\} \longmapsto S_K = \{F_p, F_s, K_p, K_s\} \tag{6.59}$$

2. Compute the minimal order $n_{\min} = n_{ellip}(F_p, F_s, K_p, K_s)$.
3. Choose the order

$$n \geq n_{\min} \tag{6.60}$$

4. Compute the selectivity factor

$$\xi = \xi_{min}\left(n, K_p, K_s\right) \tag{6.61}$$

5. Choose the ripple factor

$$\epsilon = K_p \tag{6.62}$$

6. Compute the actual passband edge

$$f_p = \frac{F_s}{\xi} \tag{6.63}$$

7. Construct the normalized lowpass transfer function

$$\mathcal{H}(n, \xi, \epsilon, p) \tag{6.64}$$

8. Construct the lowpass transfer function

$$H(s) = \mathcal{H}\left(n, \xi, \epsilon, \frac{s}{2\pi f_p}\right) \tag{6.65}$$

6.8 DESIGN D4A

1. Start from a specification and convert it into the characteristic-function-limit specification

$$\left.\begin{aligned} S_A &= \{F_p, F_s, A_p, A_s\} \\ S_\delta &= \{F_p, F_s, \delta_p, \delta_s\} \\ S_M &= \{F_p, F_s, M_p, M_s\} \\ S_r &= \{F_p, F_s, \delta_1, \delta_2\} \\ S_G &= \{F_p, F_s, G_p, G_s\} \end{aligned}\right\} \longmapsto S_K = \{F_p, F_s, K_p, K_s\} \tag{6.66}$$

2. Compute the minimal order $n_{\min} = n_{ellip}(F_p, F_s, K_p, K_s)$.
3. Choose the order

$$n \geq n_{\min} \tag{6.67}$$

4. Compute the selectivity factor,

$$\xi = \xi_{\max}\left(n, F_p, F_s, K_p, K_s\right) \tag{6.68}$$

5. Choose the ripple factor

$$\epsilon = K_p \tag{6.69}$$

6. Choose the actual passband edge

$$f_p = F_p \tag{6.70}$$

7. Construct the normalized lowpass transfer function

$$\mathcal{H}(n, \xi, \epsilon, p) \tag{6.71}$$

8. Construct the lowpass transfer function

$$H(s) = \mathcal{H}\left(n, \xi, \epsilon, \frac{s}{2\pi f_p}\right) \tag{6.72}$$

6.9 DESIGN D4B

1. Start from a specification and convert it into the characteristic-function-limit specification

$$\left.\begin{aligned} S_A &= \{F_p, F_s, A_p, A_s\} \\ S_\delta &= \{F_p, F_s, \delta_p, \delta_s\} \\ S_M &= \{F_p, F_s, M_p, M_s\} \\ S_r &= \{F_p, F_s, \delta_1, \delta_2\} \\ S_G &= \{F_p, F_s, G_p, G_s\} \end{aligned}\right\} \longmapsto S_K = \{F_p, F_s, K_p, K_s\} \tag{6.73}$$

2. Compute the minimal order $n_{\min} = n_{ellip}(F_p, F_s, K_p, K_s)$.
3. Choose the order

$$n \geq n_{\min} \tag{6.74}$$

4. Compute the maximal selectivity factor

$$\xi = \xi_{\max}(n, F_p, F_s, K_p, K_s) \tag{6.75}$$

5. Compute the ripple factor

$$\epsilon = \frac{K_s}{L(n, \xi)} \tag{6.76}$$

6. Compute the actual passband edge

$$f_p = \frac{F_s}{\xi} \tag{6.77}$$

7. Construct the normalized lowpass transfer function

$$\mathcal{H}(n, \xi, \epsilon, p) \tag{6.78}$$

8. Construct the lowpass transfer function

$$H(s) = \mathcal{H}\left(n, \xi, \epsilon, \frac{s}{2\pi f_p}\right) \tag{6.79}$$

6.10 DESIGN D5

1. Start from a specification and convert it into the characteristic-function-limit specification

$$\left.\begin{aligned} S_A &= \{F_p, F_s, A_p, A_s\} \\ S_\delta &= \{F_p, F_s, \delta_p, \delta_s\} \\ S_M &= \{F_p, F_s, M_p, M_s\} \\ S_r &= \{F_p, F_s, \delta_1, \delta_2\} \\ S_G &= \{F_p, F_s, G_p, G_s\} \end{aligned}\right\} \longmapsto S_K = \{F_p, F_s, K_p, K_s\} \tag{6.80}$$

2. Compute the minimal order $n_{\min Q}(F_p, F_s, K_p, K_s)$

3. Choose the order

$$n \geq n_{\min Q}\left(F_p, F_s, K_p, K_s\right) \tag{6.81}$$

4. Compute the maximal selectivity factor

$$\xi = \xi_{\min Q}\left(n, F_p, F_s, K_p, K_s\right) \tag{6.82}$$

and normalized frequency

$$f_{nQ}\left(n, F_p, F_s, K_p, K_s\right) \tag{6.83}$$

5. Compute the ripple factor

$$\epsilon = \frac{1}{\sqrt{L(n, \xi)}} \tag{6.84}$$

6. Compute the actual passband edge

$$f_p = \frac{F_p}{f_{nQ}\left(n, F_p, F_s, K_p, K_s\right)} \tag{6.85}$$

7. Construct the normalized lowpass transfer function

$$\mathcal{H}(n, \xi, \epsilon, p) \tag{6.86}$$

8. Construct the lowpass transfer function

$$H(s) = \mathcal{H}\left(n, \xi, \epsilon, \frac{s}{2\pi f_p}\right) \tag{6.87}$$

Summary of design parameters:

	$n_{\min}$	ξ	ϵ	f_p
D1	n_{ellip}	$\frac{F_s}{F_p}$	K_p	F_p
D2	n_{ellip}	$\frac{F_s}{F_p}$	$\frac{K_s}{L(n, \xi)}$	F_p
D3a	n_{ellip}	$\xi_{min}\left(n, K_p, K_s\right)$	K_p	F_p
D3b	n_{ellip}	$\xi_{min}\left(n, K_p, K_s\right)$	K_p	$\frac{F_s}{\xi}$
D4a	n_{ellip}	$\xi_{\max}\left(n, F_p, F_s, K_p, K_s\right)$	K_p	F_p
D4b	n_{ellip}	$\xi_{\max}\left(n, F_p, F_s, K_p, K_s\right)$	$\frac{K_s}{L(n, \xi)}$	$\frac{F_s}{\xi}$
D5	$n_{\min Q}$	$\xi_{\min Q}\left(n, F_p, F_s, K_p, K_s\right)$	$\frac{1}{\sqrt{L(n, \xi)}}$	$\frac{F_p}{f_{nQ}\left(n, F_p, F_s, K_p, K_s\right)}$

(6.88)

6.11 TIME RESPONSE AND FREQUENCY RESPONSE

From the known transfer function, $H(s)$, we compute the step response

$$h_s(t) = \mathcal{L}^{-1}\left(\frac{1}{s}H(s)\right) \tag{6.89}$$

and the impulse response

$$h(t) = \mathcal{L}^{-1}(H(s)) \tag{6.90}$$

The frequency response can be found as follows: the magnitude response

$$M(\omega) = |H(j\omega)| \tag{6.91}$$

the phase response

$$\Phi(\omega) = \arg(H(j\omega)) \tag{6.92}$$

the group delay

$$\tau_{GD}(\omega) = -\frac{\mathrm{d}\Phi(\omega)}{\mathrm{d}\omega} \tag{6.93}$$

the gain in dB

$$G(\omega) = 20\log_{10}|H(j\omega)| \tag{6.94}$$

the attenuation in dB

$$A(\omega) = -20\log_{10}|H(j\omega)| \tag{6.95}$$

6.12 HIGHPASS FILTER

A highpass filter can be specified by its edge frequencies and attenuation limits S_{Ah}: F_{ph} designates the passband edge frequency (Hz), F_{sh} is the stopband edge frequency (Hz), $F_{sh} < F_{ph}$, the passband attenuation (dB) is designated by A_{ph}, and A_{sh} stands for the stopband attenuation (dB).

First, we map the highpass filter specification, S_{Ah}, into the lowpass filter specification, S_A,

$$\boxed{S_{Ah} = \{F_{sh}, F_{ph}, A_{ph}, A_{sh}\} \longmapsto S_A = \left\{\begin{array}{l} F_p = F_{sh} \\ F_s = F_{ph} \\ A_p = A_{ph} \\ A_s = A_{sh} \end{array}\right\}} \tag{6.96}$$

Next, we construct the normalized lowpass transfer function, $\mathcal{H}(n, \xi, \epsilon, p)$, that meets the specification S_A according to the design procedures D1, D2, D3a, D3b, D4a, D4b, and D5.

Finally, the transfer function, $H_{HP}(s)$, of the highpass filter is constructed from the normalized lowpass transfer function according to the transformation:

Design	Step 6	Step 7
D1	$f_p = F_{ph}$	$p = \dfrac{2\pi f_p}{s}$
D2	$f_p = F_{ph}$	$p = \dfrac{2\pi f_p}{s}$
D3a	$f_p = F_{ph}$	$p = \dfrac{2\pi f_p}{s}$
D3b	$f_p = F_{sh}\, \xi_{min}(n, K_p, K_s)$	$p = \dfrac{2\pi f_p}{s}$
D4a	$f_p = F_{ph}$	$p = \dfrac{2\pi f_p}{s}$
D4b	$f_p = F_{sh}\, \xi_{\max}(n, F_p, F_s, K_p, K_s)$	$p = \dfrac{2\pi f_p}{s}$
D5	$f_p = F_{ph}\, f_{nQ}(n, F_p, F_s, K_p, K_s)$	$p = \dfrac{2\pi f_p}{s}$

(6.97)

which yields

$$H_{HP}(s) = \mathcal{H}\left(n, \xi, \epsilon, \frac{2\pi f_p}{s}\right) \tag{6.98}$$

6.13 BANDPASS FILTER

A bandpass filter can be specified by its edge frequencies and attenuation limits S_{Ab}; F_{p1} and F_{p2} designate the passband edge frequencies (Hz), F_{s1} and F_{s2} are the stopband edge frequencies (Hz), $F_{s1} < F_{p1} < F_{p2} < F_{s2}$, the passband attenuation (dB) is designated by A_{pb}, and A_{s1} and A_{s2} stand for the stopband attenuations (dB). We assume that the specification satisfies

$$F_{p1}F_{p2} = F_{s1}F_{s2} \tag{6.99}$$

First, we map the bandpass filter specification, S_{Ab}, into the lowpass filter specification, S_A,

$$\boxed{S_{Ab} = \{F_{s1}, F_{p1}, F_{p2}, F_{s2}, A_{s1}, A_{pb}, A_{s2}\} \longmapsto S_A = \left\{\begin{array}{l} F_p = F_{s1} \\ F_s = F_{s1}\dfrac{F_{s2} - F_{s1}}{F_{p2} - F_{p1}} \\ A_p = A_{pb} \\ A_s = \max(A_{s1}, A_{s2}) \end{array}\right\}} \tag{6.100}$$

Next, we construct the normalized lowpass transfer function, $\mathcal{H}(n, \xi, \epsilon, p)$, that meets the specification S_A according to the design procedures D1, D2, D3a, D3b, D4a, D4b, and D5.

Finally, the transfer function, $H_{BP}(s)$, of the bandpass filter is constructed from the normalized lowpass transfer function according to the transformation:

Design	Step 6	Step 7
D1	$f_p = F_{p2} - F_{p1}$	$p = \dfrac{s^2 + 4\pi^2 F_{p1} F_{p2}}{2\pi f_p s}$
D2	$f_p = F_{p2} - F_{p1}$	$p = \dfrac{s^2 + 4\pi^2 F_{p1} F_{p2}}{2\pi f_p s}$
D3a	$f_p = F_{p2} - F_{p1}$	$p = \dfrac{s^2 + 4\pi^2 F_{p1} F_{p2}}{2\pi f_p s}$
D3b	$f_p = \dfrac{\frac{F_s}{F_p}(F_{p2} - F_{p1})}{\xi_{min}\left(n, K_p, K_s\right)}$	$p = \dfrac{s^2 + 4\pi^2 F_{p1} F_{p2}}{2\pi f_p s}$
D4a	$f_p = F_{p2} - F_{p1}$	$p = \dfrac{s^2 + 4\pi^2 F_{p1} F_{p2}}{2\pi f_p s}$
D4b	$f_p = \dfrac{\frac{F_s}{F_p}(F_{p2} - F_{p1})}{\xi_{\max}\left(n, F_p, F_s, K_p, K_s\right)}$	$p = \dfrac{s^2 + 4\pi^2 F_{p1} F_{p2}}{2\pi f_p s}$
D5	$f_p = \dfrac{F_{p2} - F_{p1}}{f_{nQ}\left(n, F_p, F_s, K_p, K_s\right)}$	$p = \dfrac{s^2 + 4\pi^2 F_{p1} F_{p2}}{2\pi f_p s}$

(6.101)

which yields

$$H_{BP}(s) = \mathcal{H}\left(n, \xi, \epsilon, \frac{s^2 + 4\pi^2 F_{p1} F_{p2}}{2\pi f_p s}\right) \tag{6.102}$$

This type of bandpass filter is said to be the *symmetrical bandpass filter*.

6.14 BANDREJECT FILTER

A bandreject filter can be specified by its edge frequencies and attenuation limits S_{Ar}; F_{p1} and F_{p2} designate the passband edge frequencies (Hz), F_{s1} and F_{s2} are the stopband edge frequencies (Hz), $F_{p1} < F_{s1} < F_{s2} < F_{p2}$, the stopband attenuation (dB) is designated by A_{sr}, and A_{p1} and A_{p2} stand for the passband attenuations (dB). We assume that the specification satisfies

$$F_{p1} F_{p2} = F_{s1} F_{s2} \tag{6.103}$$

First, we map the bandreject filter specification, S_{Ar}, into the lowpass filter specification, S_A,

$$\boxed{S_{Ar} = \{F_{p1}, F_{s1}, F_{s2}, F_{p2}, A_{p1}, A_{sr}, A_{p2}\} \longmapsto S_A = \left\{\begin{array}{lcl} F_p &=& F_{p1} \\ F_s &=& F_{p1}\dfrac{F_{p2}-F_{p1}}{F_{s2}-F_{s1}} \\ A_p &=& \min(A_{p1}, A_{p2}) \\ A_s &=& A_{sr} \end{array}\right\}} \tag{6.104}$$

Next, we construct the normalized lowpass transfer function, $\mathcal{H}(n, \xi, \epsilon, p)$, that meets the specification S_A according to the design procedures D1, D2, D3a, D3b, D4a, D4b, and D5.

Finally, the transfer function, $H_{BR}(s)$, of the bandreject filter is constructed from the normalized lowpass transfer function according to the transformation:

Design	Step 6	Step 7
D1	$f_p = F_{s2} - F_{s1}$	$p = \dfrac{2\pi f_p s}{s^2 + 4\pi^2 F_{s1} F_{s2}}$
D2	$f_p = F_{s2} - F_{s1}$	$p = \dfrac{2\pi f_p s}{s^2 + 4\pi^2 F_{s1} F_{s2}}$
D3a	$f_p = F_{s2} - F_{s1}$	$p = \dfrac{2\pi f_p s}{s^2 + 4\pi^2 F_{s1} F_{s2}}$
D3b	$f_p = \dfrac{F_p}{F_s}(F_{s2} - F_{s1})\,\xi_{min}(n, K_p, K_s)$	$p = \dfrac{2\pi f_p s}{s^2 + 4\pi^2 F_{s1} F_{s2}}$
D4a	$f_p = F_{s2} - F_{s1}$	$p = \dfrac{2\pi f_p s}{s^2 + 4\pi^2 F_{s1} F_{s2}}$
D4b	$f_p = \dfrac{F_p}{F_s}(F_{s2} - F_{s1})\,\xi_{\max}(n, F_p, F_s, K_p, K_s)$	$p = \dfrac{2\pi f_p s}{s^2 + 4\pi^2 F_{s1} F_{s2}}$
D5	$f_p = (F_{s2} - F_{s1})\, f_{nQ}(n, F_p, F_s, K_p, K_s)$	$p = \dfrac{2\pi f_p s}{s^2 + 4\pi^2 F_{s1} F_{s2}}$

(6.105)

which yields

$$H_{BR}(s) = \mathcal{H}\left(n, \xi, \epsilon, \frac{2\pi f_p s}{s^2 + 4\pi^2 F_{s1} F_{s2}}\right) \tag{6.106}$$

This type of bandreject filter is said to be the *symmetrical bandreject filter*.

6.15 CONCLUDING REMARKS

We can improve the computational efficiency by using the closed-form expressions. We prefer to exploit analytical formulas involving simple algebraic manipulation, instead of using expressions in which Jacobi elliptic functions appear.